08 July 2004
JLAB-TN-04-019

Equation for the Multi-Pass Beam Breakup Threshold Current for a Single Mode and a 4x4 Recirculation Matrix

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Abstract

This paper describes a novel method for calculating the threshold of the multi-pass beam breakup (BBU) in recirculating linacs. The method is based on a simple idea of the equilibrium between power deposited by the beam in a higher-order mode (HOM) and Ohmic losses in a cavity. The method is used to derive a formula for the threshold current for a single HOM rotated at an arbitrary angle and a 4x4 recirculation matrix. We also use the method to describe HOM voltage behavior above and below the threshold current.

Introduction

A beam bunch can excite a dipole HOM in a cavity if it passes through the cavity off-axis. The magnetic field of the excited mode deflects following bunches. The kick produced by the mode can translate into a transverse displacement at the cavity location after recirculation. Thus, the recirculated beam constitutes a feedback which can cause the voltage of the HOM to grow. The equilibrium between power deposited by the beam in the HOM and the power dissipated determines the threshold current of the multi-pass beam breakup instability. Using these ideas, we derive an equation for the threshold current for a single cavity rotated at an arbitrary angle and an arbitrary 4x4 matrix in the next section. We also derive formulas that describe behavior of the HOM voltage above and below the threshold. Some applications of the threshold formula to suppression of BBU are discussed at the end of the paper.

Threshold for a Single, Rotated HOM with a 4x4 Recirculation Matrix

The voltage of a dipole HOM excited by a bunch traveling through a cavity with a displacement x is given by [1]

$$\Delta U_{in} = -qV_a \cos(\varphi)\frac{x}{a} + \frac{q}{2}V_q \frac{x}{a} \tag{1}$$

where V_a is the amplitude of the HOM voltage at the radius of the beam pipe a, and φ is its phase. The voltage V_q in the second term of equation (1) is the voltage at the beam pipe radius a induced by the bunch passing through the cavity:

$$V_q = qa^2 \frac{\omega}{2}\left(\frac{\omega}{c}\right)^2\left(\frac{R}{Q}\right)\frac{x}{a} \ . \tag{2}$$

The ½ in the second term of equation (1) reflects the beam loading theorem. The voltage V_q is much smaller than V_a for any interesting case. Thus, we can approximate the energy deposited by a bunch in a dipole HOM as

$$\Delta U = -qV_a \cos(\varphi)\frac{x}{a}. \tag{3}$$

In a two dimensional case, the transverse displacement has to be substituted with the projection of the beam displacement on the mode direction:

$$\xi = x\cos(\alpha) + y\sin(\alpha) \tag{4}$$

yielding the formula

$$\Delta U = -qV_a \cos(\varphi)\frac{\xi}{a} = -q\frac{V_a}{a}\cos(\varphi)(x\cos(\alpha) + y\sin(\alpha)) \tag{5}$$

where α is the mode rotation angle. If we assume that the last formula describes the energy deposited on the first pass, the energy deposited on the second pass, after recirculation, can be expressed as

$$\Delta U_{out} = -q\frac{V_a}{a}\cos(\varphi + \omega T_r)(x_{out}\cos(\alpha) + y_{out}\sin(\alpha)) \tag{6}$$

where we used the subscript "*out*" to mark coordinates of the recirculated bunch and assumed that the voltage amplitude V_a does not change much during the re-circulation time T_r. Coordinates of the recirculated bunch at the cavity, x_{out} and y_{out}, can be expressed in terms of the bunch coordinate at the first pass, x_{in} and y_{in}, and the transverse voltage of the HOM at the moment of the first pass as

$$\begin{aligned} x_{out} &= M_{11}x_{in} + M_{12}\frac{V_\perp\cos(\alpha)}{V_b} + M_{13}y_{in} + M_{14}\frac{V_\perp\sin(\alpha)}{V_b} \\ y_{out} &= M_{31}x_{in} + M_{32}\frac{V_\perp\cos(\alpha)}{V_b} + M_{33}y_{in} + M_{34}\frac{V_\perp\sin(\alpha)}{V_b} \end{aligned}. \tag{7}$$

In the last formula, we introduced the transverse voltage $V_\perp$ and dropped terms proportional to x'_{in} and y'_{in}. The transverse voltage can be expressed via the longitudinal voltage as [1]:

$$V_\perp = -\frac{c}{\omega}\frac{V_a}{a}\sin(\varphi). \tag{8}$$

Using the last formula, we can rewrite equation (7) as

$$\begin{aligned} x_{out} &= M_{11}x_{in} + M_{13}y_{in} - \frac{cV_a}{\omega a V_b}\sin(\varphi)\left(M_{12}\cos(\alpha) + M_{14}\sin(\alpha)\right) \\ y_{out} &= M_{31}x_{in} + M_{33}y_{in} - \frac{cV_a}{\omega a V_b}\sin(\varphi)\left(M_{32}\cos(\alpha) + M_{34}\sin(\alpha)\right) \end{aligned} . \qquad (9)$$

The average power deposited by the beam in the HOM can be calculated as the average energy deposited by individual bunches multiplied by the bunch repetition frequency:

$$\dot{U}_{beam} = \left\langle \Delta U_{in} + \Delta U_{out} \right\rangle \cdot f_b . \qquad (10)$$

The averaging is done with respect to the phase of the HOM, φ, taken at the moment of arrival of beam bunches on the first pass. The Ohmic losses in the cavity can be expressed as [1]

$$P_c = \frac{V_a^2}{a^2\left(\frac{\omega}{c}\right)^2\left(\frac{R}{Q}\right)Q} . \qquad (11)$$

Combining equations (10) and (11) the energy balance equation for the HOM has the form

$$\dot{U}_{cav} = \dot{U}_{beam} - P_c = \left\langle \Delta U_{in} + \Delta U_{out} \right\rangle \cdot f_b - P_c . \qquad (12)$$

Terms proportional to $x_{in}\cdot\cos(\varphi)$, $y_{in}\cdot\cos(\varphi)$, $x_{in}\cdot\cos(\varphi+\omega T_r)$, and $y_{in}\cdot\cos(\varphi+\omega T_r)$ yield zero after averaging if x_{in} and y_{in} are slowly varying, steering errors. If the HOM frequency is not equal to a harmonic of the bunch repetition rate, $f_b h \neq f_{hom}$, terms proportional to $\cos(\varphi)\cdot\sin(\varphi)$ yield zero and the average value of the $\sin^2(\varphi)$ term is equal to ½. Taking this into account, we can rewrite equation (12) as

$$\begin{aligned} \frac{dU}{dt} &= I_b \frac{c}{\omega}\frac{V_a^2}{V_b a^2} M_{12}^* \left\langle \sin(\varphi)\cos(\varphi + \omega T_r) \right\rangle - \frac{V_a^2}{(\omega/c)^2 a^2 (R/Q)Q} = \\ &= -I_b \frac{c}{\omega}\frac{V_a^2}{V_b a^2} M_{12}^* \frac{\sin(\omega T_r)}{2} - \frac{V_a^2}{(\omega/c)^2 a^2 (R/Q)Q} \end{aligned} \qquad (13)$$

where $M_{12}^* \equiv M_{12}\cos^2(\alpha) + (M_{14} + M_{32})\sin(\alpha)\cos(\alpha) + M_{34}\sin^2(\alpha)$.

At the threshold, $dU/dt = 0$. Thus, the threshold current is defined by

$$I_{th}\frac{M_{12}^*}{2V_b}\frac{c}{\omega}\sin(\omega T_r) + \frac{1}{(\omega/c)^2 R} = 0 \qquad (14)$$

The last equation yields the threshold current we are seeking

$$I_{th} = -\frac{2V_b}{\frac{\omega}{c}\left(\frac{R}{Q}\right) Q\, M_{12}^{*} \sin(\omega T_r)} \tag{15}$$
$$M_{12}^{*} \equiv M_{12}\cos^2(\alpha) + (M_{14} + M_{32})\sin(\alpha)\cos(\alpha) + M_{34}\sin^2(\alpha)$$

The formula is similar to the formula presented in [2], except for the modified M_{12} and the missing exponential term in the denominator, which is usually close to 1. (An alternate derivation of the above expression for the threshold current is given in Appendix A).

HOM Voltage Behavior Below and Above the Threshold

The energy, *U,* stored in the HOM can be expressed via the voltage V_a/a as

$$\frac{V_a^2}{a^2} = \omega\left(\frac{\omega}{c}\right)^2\left(\frac{R}{Q}\right) U\,. \tag{16}$$

Plugging equation (16) into equation (13), we get the equation for the energy change

$$\frac{dU}{U} = -dt\,\frac{\omega}{Q}\left(\frac{I_{th} - I}{I_{th}}\right). \tag{17}$$

The solution of the last equation is

$$U = U_0 \exp\left(-\frac{\omega}{Q}\left[\frac{I_{th} - I_b}{I_{th}}\right]\cdot t\right). \tag{18}$$

Then the HOM voltage depends on time as

$$V = V_0 \exp\left(-\frac{\omega}{2Q}\left[\frac{I_{th} - I_b}{I_{th}}\right]\cdot t\right). \tag{19}$$

Using equation (19) we can introduce the effective quality factor, Q_{eff}, for the beam-plus-HOM system, given by the equation

$$Q_{eff} = Q_0\left(\frac{I_{th}}{I_{th} - I_b}\right). \tag{20}$$

At zero current, Q_{eff} is equal to the quality factor of the cavity Q_o. As the beam current increases, Q_{eff} becomes larger and turns to infinity at the threshold. That is, the voltage in the HOM and the beam coordinate will oscillate infinitely long and will not decay. If the beam current exceeds the threshold current, the beam-plus-HOM system becomes unstable, and the amplitude of oscillations grows exponentially. The last formula can be also rewritten in terms of the decay time as:

$$\tau = \tau_0 \left(\frac{I_{th}}{I_{th} - I_b} \right). \tag{21}$$

Discussion

- Equation (15) shows that a 90° rotation inserted in the recirculation path does not necessarily eliminate the possibility of BBU. If M_{32} is not equal to $-M_{14}$ and an HOM mode is rotated at the angle α not equal to 0° or 90°, then the recirculated beam will not come back to the cavity with the angle (α + 90°) and its projection on the HOM will not be zero. To get an infinite threshold for a 4x4 matrix with zero off-diagonal 2x2 matrices and for all HOM polarizations, M_{32} has to be equal to $-M_{14}$.

- Equation (15) is correct in the non-resonant case, that is, when $f_b h \neq f_{hom}$. On resonance, the average value of the $\sin^2(\varphi)$ term is no longer equal to ½. An exact knowledge of the phase is required. The method presented in this paper does not provide exact phase information; rather it must be obtained by other means. Simulations using a newly developed 2D code show that the threshold current - while on the resonance - can be significantly different from the threshold predicted by equation (15).

- The formalism presented can be used to describe a beam-based feedback system which feeds a signal from beam position monitors back to a kicker in the injection line. If x_{in} and y_{in} vary proportionally to $V_a\cos(\omega \cdot t+\varphi_o)$, the threshold given by equation (15) will have additional terms due to the contributions of x_{in} and y_{in} which do not cancel as before (to see this, refer to the derivation in going from equation (13) to equation(15)). Care must be taken when designing such a feedback system however, since depending on the phase ($\omega \cdot t+\varphi_o$), the offending HOMs can be either damped or excited further.

- Equation (15) has been compared to results of simulations that will be presented in a separate paper [3]. The threshold predicted from equation (15) and those found from the simulations agree very well, except cases when $|\sin(\omega T_r)| << 1$. However, these cases are less interesting practically because they correspond to a very high threshold current.

Appendix A: Alternative Derivation of Threshold Current Using the Wake Potential Formalism

We present a more intuitive approach in deriving equation (15) for those readers who are more comfortable with the derivation of the threshold current using the wake potential formalism (see Refernces [2],[4]).

$$V(t) = \int_{-\infty}^{t} W(t-t')I(t'-T_r)x_{(2)}(t')\cdot dt' \tag{1'}$$

Recall that in all previous approaches, one assumes that the beam motion is decoupled, and that the orientation of the dipole HOM coincides with the *x* or *y* direction of motion. One then derives the threshold current for each plane independently. For example, if we consider the horizontal plane, we would write that the beam deflection due to the HOM voltage is given by the expression

$$\vec{r}'_o = (x'_o, 0) = \left(\frac{V_{HOM}}{V_b}, 0\right). \tag{2'}$$

The displacement on the second pass can then be written as

$$\vec{r} = (M_{12}x'_0, 0). \tag{3'}$$

However, only the component of the beam in the direction of the cavity HOM voltage can exchange energy with the mode. Thus, to find the projection of the beam displacement on the HOM voltage we take the following dot product

$$x_{(2)} = \vec{r}\cdot\hat{r}'_o = \vec{r}\cdot\frac{\vec{r}_o}{\left|\vec{r}_o\right|} = (M_{12}x'_0, 0)\cdot(1,0) = M_{12}\frac{V_{HOM}}{V_b}. \tag{4'}$$

Certainly for analysis in a single plane the vector notation used above is unnecessary. However, the purpose of the previous exercise was to show how an HOM oriented with an angle, α, with respect to the *x*-axis, and 4x4 recirculation matrix can be treated. Repeating the procedure above, equations (2'), (3') and (4') become

$$\vec{r}'_o = (x'_o, y'_o) = \frac{V_{HOM}}{V_b}(\cos(\alpha), \sin(\alpha)) \tag{5'}$$

$$\vec{r} = (M_{12}x'_0 + M_{14}y'_o, M_{32}x'_o M_{34}y'_o) \tag{6'}$$

$$\vec{r}\cdot\hat{r}_o' = \vec{r}\cdot\frac{\vec{r}_o'}{\left|\vec{r}_o'\right|} = \frac{V_{HOM}}{V_b}\left(M_{12}\cos^2(\alpha) + (M_{14}+M_{32})\sin(\alpha)\cos(\alpha) + M_{34}\sin^2(\alpha)\right) \qquad (7')$$

One can use the usual methods and use the result of equation (7’) in equation (1’) to extract the threshold current. Suffice it to say that the result will be exactly the expression given in equation (15).

References

[1] H. Padamsee, J. Knobloch, T. Hays, "RF Superconductivity for Accelerators", Wiley and Sons (1998).

[2] N. Sereno, "Experimental studies of multipass beam breakup and energy recovery using the CEBAF injector linac", Ph.D. Thesis, U. of Illinois at Urbana-Champaign, (1994).

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[4] Krafft, G., Bisognano, J., Laubach, S., "Calculating Beam Breakup in Superconducting Linear Accelerators", *Unpublished,* (1991).