Measurement Formulae for a Superconducting RF Cavity without Beam Load

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Abstract

The superconductivity in Radio Frequency (RF) is an increasingly important branch of accelerator physics and technology because it heightens accelerating performances and lowers operating expenses. The normal method for estimating a superconducting RF (SRF) cavity performance is through low power and high power measurements without a beam load. At present, the one-port SRF cavity measurement equations are the most popular formulae of the SRF cavity measurements. The one-port SRF cavity measurement equations are the approximation of the two-port measurement equations. To understand cavity behavior and performance more accurately, in this paper, the two-port SRF cavity's measurement equations for low power and high power measurements are developed at steady state and transient state by a series equivalent circuit. As an example, the modified measurement equations are deduced for the exponential decayed incident power pulse.

I. Introduction

RF superconductivity is an increasingly important branch of accelerator physics and technology because of its high accelerating performances and low operating expenses. Accelerator projects are increasingly based on Superconducting RF technology, e.g. storage rings (CERN, DESY, KEK), linacs for electrons or positrons (Frascati, JLab, HEPL, Darmstadt), linacs for heavy ions (Argonne, Legnaro, JAERI), the Spallation Neutron Source (ORNL), and the Energy Recovery Linacs (ERLs) (JLab-FEL). Development programs are underway for high current ERLs (Cornell, KEK, JLab) which may open up a whole new opportunities for industrial and research applications. At present, the high energy physicists and accelerator experts come to an agreement that the next high energy International Linear Collider (ILC) should use Superconducting RF technology.

The low power and high power measurements without a beam load are necessary methods of estimating superconducting RF cavity performance before the SRF cavity operation of accelerating beam. Padamsee etc. have developed the one-port, namely only Fundamental Power Coupler (FPC), SRF cavity's measurement equations [1] by using cavity stored energy

change in a parallel equivalent circuit. J. Weaver,[2] G. A. Loew[3] and L. Merminga[4] obtained the one-port SRF cavity's equations at steady state and transient state using a parallel equivalent circuit and the Laplace transformations. At present, the one-port SRF cavity's measurement equations are the most popular formulae of the SRF cavity measurements.

The input power coupler is normally used for the one-port measurement. In the conditions of a pulse incident power, the measurement signals are incident and reflected and transmitted signals. The stability of the phase and amplitude flatness on the top of the accelerating field pulse in the SRF cavity are very important for pulsed accelerating beam bunches. The reflected wave frequency is not as same as the cavity's, and its amplitude has no direct and clear relation with cavity field's amplitude. So a reflected signal can not be directly used to control the phase and the flatness of the pulse, especially for a short pulse. In contrast, the second port, for example, the field probe (FP) signal has a same frequency and a direct relation with the cavity. In reality, most superconducting cavities have more than two ports. For example, the Spallation Neutron Source (SNS) cavities have four ports: one FPC, one FP, and two Higher Order Mode couplers (HOMs). Developing two-port cavity's measurement equations is necessary to understand cavity behavior and to control cavity's phase and amplitude more accurately for SRF cavity's measurements and beam operation.

In this paper, a series equivalent circuit is used to develop two-port cavity measurement equations. First, the cavity equivalent circuit model was set up, and the relation between cavity's and circuit's parameters is introduced. Then the cavity's equations at steady state are derived. These equations are used to do lower power measurements (<1 Watt), Continuous Wave (CW) high power measurements, and the cavity's port parameter calibration. After that, the transient state equations are obtained for a square wave drive pulse by solving differential equation. These equations are used in the pulsed incident high power measurements. For non-square incident wave, the measurement formulae need to be modified. As an example, the modified equations are obtained for an exponential decayed and pulsed incident power.

II. Equivalent Circuit Model.

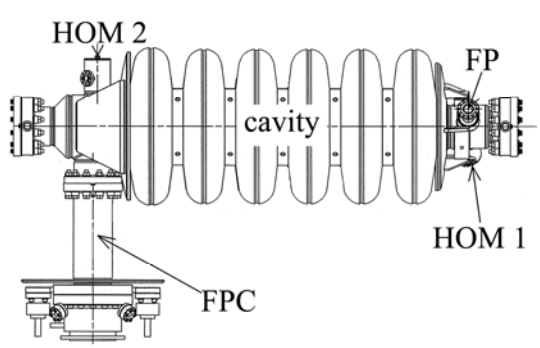
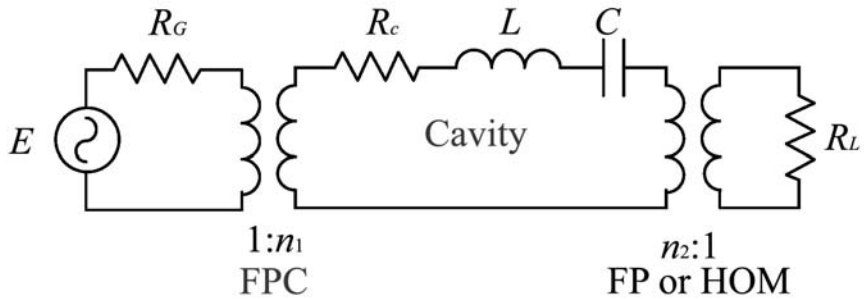
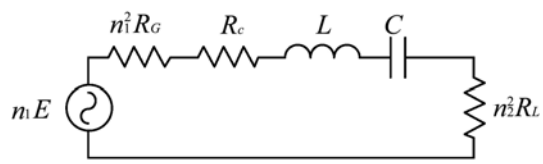


Fig. 1 SNS medium β (0.61) cavity structure.

In general, a superconducting cavity has one FPC, one FP, and two or more HOMs. FPC supplies the fundamental RF power to the cavity. The RF signal picked up from the FP is used to detect the field phase and gradient in the cavity. HOMs are used to damp the HOM power. Fig.1 is a SNS 6-cell cavity with four ports.



(a) Equivalent circuit of a two-port cavity coupling system.



(b) Alternative form of the circuit (a) without transformers.

Fig.2 Equivalent circuit of a two-port cavity coupling system at the reference planes of cavity side.

In the two-port measurements, the signals that we can measure are incident, reflected, and transmitted. The emitted signal is defined as the reflected signal from the stored energy in the system when the incident signal is switched off. The cavity can be equivalent to either a parallel or a series resonance circuit. The series circuit is comparatively simple to develop for two-port cavity equations without beam loading. The cavity RF characteristics can be derived from this series circuit. Assuming the impedances of signal source and load are real and given by R_G and R_L, as shown in Fig2, where R_c, L, C are the resistance, inductance and capacitance of the SRF cavity. E is the equivalent voltage of generator in frequency ω. For a high frequency generator, the harmonic time structure is much smaller than the amplitude modulation by scale of several orders (10^{-6} for SNS case). So the voltage amplitude can be written as $E(\omega,t)=E(t)\exp(i\omega t)$. then the circuit equivalent current is $I(\omega,t)=I(t)\exp(i\omega t)$. The cavity voltage V_c 's amplitude is

$$\left|V_c(\omega,t)\right| = \frac{\left|I(\omega,t)\right|}{\omega C} \tag{1}$$

The cavity voltage V_c can be related to the maximum accelerating gradient *Eacc* when the beam bunch is accelerated at the

RF wave crest, and related to effective accelerating length d as well as the Transit Time Factor (TTF) T_T.

$$E_{acc}(\omega) = \frac{|V_c(\omega, t_p)| T_T}{d}$$

The t_p is at the flat top the RF pulse. Normally, the cavity's intrinsic parameters are defined at the cavity's resonance frequency ω_0. The cavity stored energy U at ω_0 is:

$$U(\omega_0, t) = L|I(\omega_0, t)|^2 = C|V_c(\omega_0, t)|^2 = \frac{|I(\omega_0, t)|^2}{\omega_0{}^2 C}$$

The cavity's emitted power P_e, dissipated power P_d and transmitted power P_t are:

$$\begin{cases} P_e(\omega_0, t) = n_1^2 R_G |I(\omega_0, t)|^2 \\ P_d(\omega_0, t) = R_c |I(\omega_0, t)|^2 \\ P_t(\omega_0, t) = n_2^2 R_L |I(\omega_0, t)|^2 \end{cases} \tag{2}$$

The cavity's intrinsic quality factor is cavity geometry and surface resistance dependent only:

$$Q_0 = \omega_0 U(\omega_0, t) / P_d(\omega_0, t) = \omega_0 L / R_c = 1/(\omega_0 C R_c) \quad \text{with} \quad \omega_0 = \sqrt{1/(LC)} \tag{3}$$

The external quality factor of the cavity input port is cavity and coupler's geometry dependent only:

$$Q_e = \omega_0 U(\omega_0, t) / P_e(\omega_0, t) = \omega_0 L / (n_1^2 R_G) = 1/(\omega_0 C n_1^2 R_G) \tag{4}$$

The external quality factor of the cavity output port is similar:

$$Q_t = \omega_0 U(\omega_0, t) / P_t(\omega_0, t) = \omega_0 L / (n_2^2 R_L) = 1/(\omega_0 C n_2^2 R_L) \tag{5}$$

The coupling coefficients of the input port and output port to cavity are also cavity surface resistance dependent:

$$\begin{cases} \beta_e = Q_0 / Q_e = P_e / P_d = n_1^2 R_G / R_c \\ \beta_t = Q_0 / Q_t = P_t / P_d = n_2^2 R_L / R_c \end{cases} \tag{6}$$

The loaded Q_L is:

$$Q_L = \frac{\omega_0 L}{R_c + n_1^2 R_G + n_2^2 R_L} = \frac{Q_0}{1 + \beta_e + \beta_t}$$

or

$$\frac{1}{Q_L} = \frac{R_c + n_1^2 R_G + n_2^2 R_L}{\omega_0 L} = \frac{1}{Q_0} + \frac{1}{Q_e} + \frac{1}{Q_t} = \frac{1}{Q_0}(1 + \beta_e + \beta_t) \tag{7}$$

The cavity shunt impedance R is:

$$R = \frac{|V_c|^2}{P_d} = \frac{1}{R_c \omega_0^2 C^2} \quad \text{and} \quad R / Q_0 = \frac{1}{\omega_0 C} = \omega_0 L \tag{8}$$

Normally the R/Q_0 is obtained by a cavity simulation code and written as R/Q. Please pay attention that, this R/Q is still not yet used by accelerator physicist or a linac's definition which should be:

$$R_{acc}/Q = \frac{|V_c|^2 T_T^{\,2}}{\omega_0 U(\omega_0, t_p)} = (R/Q_0)T_T^{\,2} \quad \text{or} \quad = (R/Q)T_T^{\,2}$$

Based on above equations, the cavity equivalent circuit parameters can be expressed with the cavity parameters as:

$$\begin{cases} C = \dfrac{1}{(R/Q)\omega_0} \\ L = \dfrac{(R/Q)}{\omega_0} \\ R_c = \dfrac{(R/Q)}{Q_0} = \dfrac{(R/Q)}{G} \cdot R_s \end{cases} \tag{9}$$

Note that G here is cavity geometry factor in the unit of resistance. R_c here is not as same as the conventional series resonance circuit where the R_s denotes as the cavity surface resistance.

Table 1 lists the SNS medium beta cavity parameters and its corresponding equivalent circuit parameters.

Table 1. A SNS medium beta cavity parameters and its equivalent circuit parameters.

SNS medium beta cavity parameters		Equivalent circuit parameters	
f_0	805 MHz	L	119.41 nH
R_{acc}/Q	279 Ω	C	0.3273pF
Q_e	7.3E+5	n_1	4.07E-3
Q_t	1.26E+12	n_2	3.10E-6
Q_0	8.85E+9	R_c	68.2 nΩ
d	0.682 m	R_L=R_G	50 Ω
T_T	0.679		
R/Q	604 Ω		
G	177 Ω		
R_s	20 nΩ		

For the circuit shown in Figure 2 (b), its differential equation relating to the current $I(\omega,t)$, and voltage $E(\omega,t)$ is:

$$L\frac{dI(\omega,t)}{dt} + (n_1^2 R_G + R_c + n_2^2 R_L)I(\omega,t) + \frac{1}{C}\int I(\omega,t)dt = n_1 E(\omega,t) \tag{10}$$

This is the SRF cavity's basic equation based on the series equivalent circuit. In the following sections, we will discuss the solution of this equation for different driving signal $E(\omega,t)$.

III. Steady State Equations

If an incident power is Continuous Wave (CW), and its amplitude change reaches a constant after charging time of $10Q_L/\omega_0$. At the steady state, the generator voltage $E(\omega,t)=E_0\exp(i\omega t)$ and the current $I(\omega,t)=I_0\exp(i\omega t)$. Substituting these into equation (10), we find

$$L\frac{d^2I(\omega,t)}{dt^2}+R_T\frac{dI(\omega,t)}{dt}+\frac{I(\omega,t)}{C}=i\omega n_1E_0\exp(i\omega t) \tag{11}$$

Here $R_T=n_1^2R_G+R_1+n_2^2R_L$. The characteristic number $-\lambda^2$ of the above equation is:

$$\lambda^2=\frac{4L}{C}-R_T{}^2=4R_c^2Q_0^2\left[1-\frac{1}{4Q_L^2}\right]=4\varepsilon^2R_c^2Q_0^2>0. \tag{12}$$

in which $\varepsilon=\sqrt{1-1/(4Q_L^2)}$. The solution of the equation (11) is:

$$I(\omega,t)=\frac{n_1E_0}{R_T+i\omega L+\frac{1}{i\omega C}}\left\{\exp(i\omega t)+A\exp\left[-\frac{R_T}{2L}t-i\left(\omega+\frac{\lambda}{2L}\right)t\right]+B\exp\left[-\frac{R_T}{2L}t-i\left(\omega-\frac{\lambda}{2L}\right)t\right]\right\} \tag{13}$$

Because the steady state current intensity $|I(\omega,t)|^2$ is a constant, the $A=B=0$ in equation (13). Then the circuit current intensity $I(\omega,t)$ can be written as:

$$I(\omega,t)=\frac{n_1E(\omega,t)}{R_c\left[(1+\beta_e+\beta_t)+iQ_0\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\right]}. \tag{14}$$

And the modulus of the current intensity $I(\omega,t)$ is:

$$|I(\omega,t)|=\sqrt{I(\omega,t)I^*(\omega,t)}=\frac{n_1|E(\omega,t)|}{R_c(1+\beta_e+\beta_t)}\frac{1}{\sqrt{1+Q_L^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}} \tag{15}$$

Above equation shows the cavity circuit current $I(\omega,t)$ is decided not only by the incident power amplitude and cavity intrinsic parameters (such as surface resistance, coupling factors), but also the frequency deference between incident power (klystron) frequency ω and cavity resonance frequency ω_0. The incident power from generator is: $P_{in}=|E(\omega,t)|^2/4R_G$ [5].

For a CW operation, we can ignore the time dependent part at this moment. The cavity dissipated power is:

$$P_d(\omega) = R_c \left| I(\omega,t) \right|^2 = \frac{4\beta_e}{(1+\beta_e+\beta_t)^2 \left[1 + Q_L^2 \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)^2 \right]} P_{in} \tag{16}$$

The transmitted power P_t is written as

$$P_t(\omega) = n_2^2 R_L \left| I(\omega,t) \right|^2 = \frac{4\beta_e\beta_t}{(1+\beta_e+\beta_t)^2 \left[1 + Q_L^2 \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)^2 \right]} P_{in} = \beta_t P_d(\omega) \tag{17}$$

The transmission coefficient T can written as

$$T(\omega) = \frac{P_t(\omega)}{P_{in}} = \frac{4\beta_e\beta_t}{(1+\beta_e+\beta_t)^2 \left[1 + Q_L^2 \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)^2 \right]} \tag{18}$$

Normally, scattering transmission parameter S_{21} is used to measure the $T(\omega)$. The $T(\omega)$ can be written as:

$$\mathrm{S}_{21}(\omega) = 10\log T(\omega) = 10\log \left\{ \frac{4\beta_e\beta_t}{(1+\beta_e+\beta_t)^2 \left[1 + Q_L^2 \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)^2 \right]} \right\} \text{ (dB)} \tag{19}$$

At cavity resonance frequency:

$$\mathrm{S}_{21}(\omega_0) = 10\log \left[\frac{4\beta_e\beta_t}{(1+\beta_e+\beta_t)^2} \right] \text{(dB)} \tag{20}$$

If $\Delta\omega_b$ is the cavity bandwidth, namely $Q_L = \omega_0 / \Delta\omega_b$, putting $\omega = \omega_0 \pm \Delta\omega_b/2$, then for $Q_L >> 1$ or $\omega_0 >> \Delta\omega$

$$T(\omega_0 \pm \Delta\omega_b / 2) \approx T(\omega_0)/2$$

or

$$\mathrm{S}_{21}(\omega_0 \pm \Delta\omega_b / 2) \approx 10\log T(\omega_0) - 3 \text{ (dB).} \tag{21}$$

This equation is used to measure cavity Q_L in low power measurement, for example, by a network analyzer. The cavity emitted power P_e is:

$$P_e(\omega) = n_1^2 R_G \left| I(\omega,t) \right|^2 = \frac{4\beta_e^2}{(1+\beta_e+\beta_t)^2 \left[1 + Q_L^2 \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)^2 \right]} P_{in} = \beta_e P_d(\omega) \tag{22}$$

This emitted power in a CW operation has no physical meaning when the RF generator is the power source. Because we are

transforming the RF circulator load R_G into the cavity side of the resistance $n_1^2 R_G$. When the power source is switched off, we will see from the next section that, the emitted power will replace the reflected power. Then the cavity's stored energy is the power source. The circulator resistance R_G is the heat load for the emitted power through the FPC transformer n_1.

From equation (16), (17) and (22), the stored energy becomes:

$$Q_0 P_d(\omega) = Q_e P_e(\omega) = Q_t P_t(\omega) = \omega U(\omega) = \frac{4\beta_e Q_L}{(1+\beta_e+\beta_t)\left[1+Q_L^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2\right]} P_{in} \tag{23}$$

Then the cavity accelerating gradient can be written as:

$$\begin{aligned} E_{acc}(\omega) &= \frac{|I(\omega,t)|T_T}{d\omega C} = \sqrt{\frac{4\beta_e (R_{acc}/Q) Q_L P_{in}}{d^2(1+\beta_e+\beta_t)\frac{\omega^2}{\omega_0^2}\left[1+Q_L^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2\right]}} \\ &= \sqrt{\frac{(R_{acc}/Q)\omega_0^2 Q_0 P_d(\omega)}{d^2\omega^2}} = \sqrt{\frac{(R_{acc}/Q)\omega_0^2 Q_e P_e(\omega)}{d^2\omega^2}} = \sqrt{\frac{(R_{acc}/Q)\omega_0^2 Q_t P_t(\omega)}{d^2\omega^2}} \end{aligned} \tag{24}$$

For $\omega=\omega_0$, equation (21) can be written as

$$E_{acc}(\omega_0) = \sqrt{\frac{4\beta_e (R_{acc}/Q) Q_L P_{in}}{d^2(1+\beta_e+\beta_t)}} = \sqrt{\frac{(R_{acc}/Q) Q_0 P_d(\omega_0)}{d^2}} = \sqrt{\frac{(R_{acc}/Q) Q_e P_e(\omega_0)}{d^2}} = \sqrt{\frac{(R_{acc}/Q) Q_t P_t(\omega_0)}{d^2}} \tag{25}$$

The reflected power P_r is:

$$P_r(\omega) = P_{in} - P_d(\omega) - P_t(\omega) = \frac{(1-\beta_e+\beta_t)^2 + Q_0^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}{(1+\beta_e+\beta_t)^2 + Q_0^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2} P_{in} \tag{26}$$

According to above equation, the reflection coefficient $\Gamma_e(\omega)$ for the input port can be written as

$$|\Gamma_e(\omega)|^2 = \frac{P_r}{P_{in}} = \frac{(1-\beta_e+\beta_t)^2 + Q_0^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}{(1+\beta_e+\beta_t)^2 + Q_0^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2} \tag{27}$$

For $\omega=\omega_0$,

$$\Gamma_e(\omega_0) = \frac{|1-\beta_e+\beta_t|}{1+\beta_e+\beta_t} = \mathrm{S}_{11} \tag{28}$$

The reflection coefficient $\Gamma_t(\omega_0)$ of the output port is by switching "t" and "e" notes:

$$\Gamma_t(\omega_0) = \frac{|1+\beta_e-\beta_t|}{1+\beta_e+\beta_t} = \mathrm{S}_{22} \tag{29}$$

From equation (28) and (29), the FPC coupling factor β_e and FP coupling factor β_t can be expressed as:

$$\begin{cases} \beta_e = \dfrac{1+\Gamma_e(\omega_0)}{\Gamma_t(\omega_0)-\Gamma_e(\omega_0)} \\ \beta_t = \dfrac{1-\Gamma_t(\omega_0)}{\Gamma_t(\omega_0)-\Gamma_e(\omega_0)} \end{cases} \text{(for } \beta_e \geq \beta_t+1) \tag{30}$$

A typical design for FPC is over-coupled (β_e>1), and the FP under-coupled (β_t<1). Above equation can be used as long as $\beta_e \geq \beta_t+1$. At this situation, the FP reflection coefficient S_{22} is less than FPC reflection coefficient S_{11}.

For other condition like when testing a SRF cavity vertically where the input power is limited, the FPC coupling factor β_e could be closed to 1 and FP coupling factor β_t <<1:

$$\begin{cases} \beta_e = \dfrac{1-\Gamma_e(\omega_0)}{\Gamma_t(\omega_0)+\Gamma_e(\omega_0)} \\ \beta_t = \dfrac{1-\Gamma_t(\omega_0)}{\Gamma_t(\omega_0)+\Gamma_e(\omega_0)} \end{cases} \text{(for} |\beta_e-\beta_t| \leq 1\text{).} \tag{31}$$

Using above results and equation (21), the cavity FPC external quality factor Q_e and the FP external quality factor Q_t are:

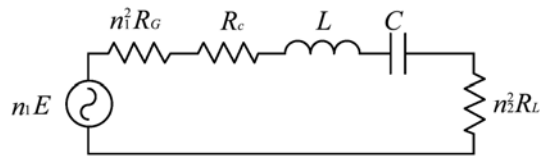
$$\begin{cases} Q_e = \dfrac{4\beta_t}{1+\beta_e+\beta_t} \cdot \dfrac{Q_L}{T(\omega_0)} = \dfrac{4\beta_t}{1+\beta_e+\beta_t} Q_L 10^{-0.1\mathrm{S}_{21}(dB)} \\ Q_t = \dfrac{4\beta_e}{1+\beta_e+\beta_t} \cdot \dfrac{Q_L}{T(\omega_0)} = \dfrac{4\beta_e}{1+\beta_e+\beta_t} Q_L 10^{-0.1\mathrm{S}_{21}(dB)} \end{cases} \tag{32}$$

Equations (30) to (32) are used to measure and calibrate FPC's and FP's external Q at warm cavity or during the prototyping to measure the external Q of HOM couplers. The cavity's Q_0 can be obtained by equation (7) then, but the error would not be acceptable for a cold (2K) cavity when β_e>>1.

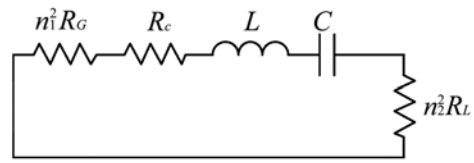
IV. Transient State Equations

If the incident power is pulsed wave, the cavity is in a transient state at the pulse start and end. As shown in the equivalent circuit Fig 3, the pulse start is defined as RF switch on; and the pulse end is RF switch off. First we consider the pulse is a

square wave. For a standard square wave pulse, $E(\omega,t)=E_0\exp(i\omega t)$ after switching RF on, and $E(\omega,t)=0$ after switching RF off.



a. RF Switch on.



b. RF Switch off.

Fig. 3 The equivalent circuits in transient state.

A. RF Switch On

For RF switch on, substituting $E(\omega,t)=E_0\exp(i\omega t)$ into equation (10), at steady state, the circuit current $I(\omega,\mathrm{t})$ can be obtained as equation (13).

For most cavities, their loaded quality factor Q_L is more than 1000, even at room temperature. This means that the 1-ε is less than 1.25E-7. So, the $\lambda/2L$ can be approximately replaced by ω_0. Then we have $\omega\pm\lambda/2L=\omega\pm\omega_0$. The circuit current amplitude related to the cavity voltage is a slow variation comparing to cavity frequency ω_0. So an approximation [6] can be used to simplify equation (13). The boundary condition of the switch on is: $I(\omega,0)=0$ and $I(\omega,t\to+\infty)=Constant$. Noting that $R_T/L=\omega_0/Q_L$, the circuit current $I(\omega,t)$ during RF switch on becomes:

$$I(\omega,t)=\frac{n_1E_0}{R_c\left[(1+\beta_e+\beta_t)+iQ_0\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)\right]}\left\{1-\exp\left[-\frac{\omega_0}{2Q_L}t-i\Delta\omega t\right]\right\}\exp(i\omega t) \tag{33}$$

It's corresponding current module square $|I(\omega,t)|^2$ is

$$|I(\omega,t)|^2=\frac{4\beta_e}{R_c(1+\beta_e+\beta_t)^2\left[1+Q_L^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2\right]}\left\{1+\exp\left(-\frac{\omega_0}{Q_L}t\right)-2\exp\left(-\frac{\omega_0}{2Q_L}t\right)\cos(\Delta\omega t)\right\}P_{in}. \tag{34}$$

Here $\Delta\omega=\omega-\omega_0$. Based on this current, the cavity dissipated power $P_d(\omega,t)$ is:

$$P_d(\omega,t)=R_c\left|I(\omega,t)\right|^2=\frac{4\beta_e\left[1+\exp\left(-\frac{\omega_0}{Q_L}t\right)-2\exp\left(-\frac{\omega_0}{2Q_L}t\right)\cos(\Delta\omega t)\right]}{(1+\beta_e+\beta_t)^2\left[1+Q_L^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2\right]}P_{in} \tag{35}$$

Transmitted power P_t is written as:

$$P_t(\omega,t)=n_2^2R_L\left|I(\omega,t)\right|^2=\frac{4\beta_e\beta_t\left[1+\exp\left(-\frac{\omega_0}{Q_L}t\right)-2\exp\left(-\frac{\omega_0}{2Q_L}t\right)\cos(\Delta\omega t)\right]}{(1+\beta_e+\beta_t)^2\left[1+Q_L^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2\right]}P_{in}=\beta_tP_d(\omega,t) \tag{36}$$

The transmission coefficient $T(\omega, t)$ can be written as:

$$T(\omega,t)=\frac{P_t(\omega,t)}{P_{in}(\omega,t)}=\frac{4\beta_e\beta_t\left[1+\exp\left(-\frac{\omega_0}{Q_L}t\right)-2\exp\left(-\frac{\omega_0}{2Q_L}t\right)\cos(\Delta\omega t)\right]}{(1+\beta_e+\beta_t)^2\left[1+Q_L^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2\right]} \tag{37}$$

and

$$T(\omega_0,t)=\frac{4\beta_e\beta_t}{(1+\beta_e+\beta_t)^2}\left[1-\exp\left(-\frac{\omega_0}{2Q_L}t\right)\right]^2 \tag{38}$$

The cavity emitted power P_e in this case has no physical meaning but relation to P_d and P_t:

$$P_e(\omega,t)=n_1^2R_G\left|I(\omega,t)\right|^2=\frac{4\beta_e^2\left\{1+\exp\left(-\frac{\omega_0}{Q_L}t\right)-2\exp\left(-\frac{\omega_0}{2Q_L}t\right)\cos(\Delta\omega t)\right\}}{(1+\beta_e+\beta_t)^2\left[1+Q_L^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2\right]}P_{in}=\beta_eP_d(\omega,t)=\frac{\beta_e}{\beta_t}P_t(\omega,t) \tag{39}$$

The equation (35), (36) and (39) display the $P_d(t)$, $P_t(t)$ and $P_e(t)$ have similar relation at pulse mode as equation (23):

$$\omega U(\omega,t)=Q_0P_d(\omega,t)=Q_eP(\omega,t)_e=Q_tP_t(\omega,t)=\frac{4\beta_eQ_L\left[1+\exp\left(-\frac{\omega_0}{Q_L}t\right)-2\exp\left(-\frac{\omega_0}{2Q_L}t\right)\cos(\Delta\omega t)\right]}{(1+\beta_e+\beta_t)\left[1+Q_L^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2\right]}P_{in} \tag{40}$$

Then the cavity accelerating gradient can be written as:

$$E_{acc}(\omega,t)=\sqrt{\frac{4\beta_e(R_{acc}/Q)Q_LP_{in}}{d^2(1+\beta_e+\beta_t)\frac{\omega^2}{\omega_0^2}\left[1+Q_L^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2\right]}\left[1+\exp\left(-\frac{\omega_0}{Q_L}t\right)-2\exp\left(-\frac{\omega_0}{2Q_L}t\right)\cos(\Delta\omega t)\right]}$$
$$=\frac{\omega_0}{d\omega}\sqrt{(R_{acc}/Q)Q_0P_d(t)}=\frac{\omega_0}{d\omega}\sqrt{(R_{acc}/Q)Q_eP_e(t)}=\frac{\omega_0}{d\omega}\sqrt{(R_{acc}/Q)Q_tP_t(t)} \tag{41}$$

For $\omega=\omega_0$, equation (41) can be rewritten as:

$$E_{acc}(\omega_0,t)=\sqrt{\frac{4\beta_e(R_{acc}/Q)Q_LP_{in}(t)}{d^2(1+\beta_e+\beta_t)}\left[1-\exp\left(-\frac{\omega_0}{2Q_L}t\right)\right]}$$
$$=\frac{1}{d}\sqrt{(R_{acc}/Q)Q_0P_d(t)}=\frac{1}{d}\sqrt{(R_{acc}/Q)Q_eP_e(t)}=\frac{1}{d}\sqrt{(R_{acc}/Q)Q_tP_t(t)} \tag{42}$$

The cavity's stored energy change is:

$$\frac{dU}{dt}(\omega,t)=\frac{d}{dt}\left(\frac{1}{2}L\left|I(\omega,t)\right|^2+\frac{1}{2}\frac{\left|I(\omega,t)\right|^2}{\omega^2C}\right)=\frac{L}{2}\left(1+\frac{\omega_0^2}{\omega^2}\right)\frac{d\left|I(\omega,t)\right|^2}{dt} \tag{43}$$

Because

$$\frac{d\left|I(\omega,t)\right|^2}{dt}=\frac{4\beta_e\exp\left(-\frac{\omega_0}{2Q_L}t\right)}{R_c\left\{\left(1+\beta_e+\beta_t\right)^2+Q_0^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2\right\}}\left[\frac{\omega_0}{Q_L}\cos(\Delta\omega t)-\frac{\omega_0}{Q_L}\exp\left(-\frac{\omega_0}{2Q_L}t\right)+2\Delta\omega\sin(\Delta\omega t)\right]P_{in}$$

the dU/dt can be written as:

$$\frac{dU}{dt}(\omega,t)=\frac{2Q_0\beta_e\left(1+\frac{\omega_0^2}{\omega^2}\right)\exp\left(-\frac{\omega_0}{2Q_L}t\right)}{\left(1+\beta_e+\beta_t\right)^2+Q_0^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}\left[\frac{1}{Q_L}\cos(\Delta\omega t)-\frac{1}{Q_L}\exp\left(-\frac{\omega_0}{2Q_L}t\right)+2\frac{\Delta\omega}{\omega_0}\sin(\Delta\omega t)\right]P_{in} \tag{44}$$

The cavity stored energy $U(t)$ is:

$$U(\omega,t)=\int_0^t \frac{dU}{d\tau}(\omega,\tau)d\tau=\frac{2\beta_e\left(1+\frac{\omega_0^2}{\omega^2}\right)Q_L}{\omega_0(1+\beta_e+\beta_t)\left[1+Q_L^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2\right]}\left\{1+\exp\left(-\frac{\omega_0}{Q_L}t\right)-2\exp\left(-\frac{\omega_0}{2Q_L}t\right)\cos(\Delta\omega t)\right\}P_{in}$$

$$=\left(1+\frac{\omega_0^2}{\omega^2}\right)\frac{Q_L(1+\beta_e+\beta_t)}{2\omega_0}P_d(\omega,t) \tag{45}$$

The cavity intrinsic quality is:

$$Q_0(\omega)=\frac{\omega U(\omega,t)}{P_d(\omega,t)}=\frac{\omega}{2\omega_0}\left(1+\frac{\omega_0^2}{\omega^2}\right)Q_L(1+\beta_e+\beta_t) \tag{46}$$

Comparing equation (46) and equation (7), we find there is a different factor of $\frac{\omega}{2\omega_0}\left(1+\frac{\omega_0^2}{\omega^2}\right)$. For $\omega=\omega_0$, the equation (46) becomes (7).

The reflected power P_r is:

$$P_r(t)=P_{in}-P_d(t)-\frac{dU}{dt}(t)-P_t(t)$$

For $\omega=\omega_0$, the reflected power could become as:

$$P_r=P_{in}\left\{\begin{array}{l}1-\frac{4\beta_e}{(1+\beta_e+\beta_t)^2}\left[1-\exp\left(-\frac{\omega_0}{2Q_L}t\right)\right]^2-\frac{4\beta_e}{1+\beta_e+\beta_t}\exp\left(-\frac{\omega_0}{2Q_L}t\right)\left[1-\exp\left(-\frac{\omega_0}{Q_L}t\right)\right]\\ -\frac{4\beta_e\beta_t}{(1+\beta_e+\beta_t)^2}\left[1-\exp\left(-\frac{\omega_0}{2Q_L}t\right)\right]^2\end{array}\right\} \tag{47}$$

This equation can be simplified as following and is similar to the one-port equation except a β_t at the bottom.

$$P_r=\left\{1-\frac{2\beta_e}{1+\beta_e+\beta_t}\left[1-\exp\left(-\frac{\omega_0}{2Q_L}t\right)\right]\right\}^2 P_{in} \tag{48}$$

B. RF Switch Off

After switch RF off, namely, after a pulse length of τ_0, $E(\omega,t)=0$, the equation (11) becomes as:

$$L\frac{d^2I(\omega,t)}{dt^2}+R_T\frac{dI(\omega,t)}{dt}+\frac{I(\omega,t)}{C}=0\,. \tag{49}$$

As shown in Fig. 3 (b), the circuit frequency is dominated by cavity frequency ω_0 after RF switch off. The initial boundary

condition of the switch off is:

$$I(\omega_0,\tau_0)=\frac{n_1E_0\exp(i\omega_0\tau_0)}{R_c(1+\beta_e+\beta_t)}\left\{1-\exp\left(-\frac{\omega_0}{2Q_L}\tau_0\right)\right\} \tag{50}$$

and $I(\omega_0,t\to+\infty)=0$.

Using this boundary condition, the circuit current $I(\omega_0,t)$ can be solved:

$$I(\omega_0,t)=I(\omega_0,\tau_0)\exp\left[-\frac{\omega_0}{2Q_L}(t-\tau_0)-i\omega_0(t-\tau_0)\right] \tag{51}$$

And the circuit current module square $|I(\omega,t)|^2$ is:

$$|I(\omega_0,t)|^2=|I(\omega_0,\tau_0)|^2\exp\left[-\frac{\omega_0}{Q_L}(t-\tau_0)\right] \tag{52}$$

The cavity's dissipated power P_d, the transmitted power P_t and the emitted power P_e are:

$$\begin{cases} P_d(t)=P_d(\omega_0,\tau_0)\exp\left[-\frac{\omega_0}{Q_L}(t-\tau_0)\right] \\ P_t(t)=P_t(\omega_0,\tau_0)\exp\left[-\frac{\omega_0}{Q_L}(t-\tau_0)\right]=\beta_tP_d(t) \\ P_e(t)=P_e(\omega_0,\tau_0)\exp\left[-\frac{\omega_0}{Q_L}(t-\tau_0)\right]=\beta_eP_d(t) \end{cases} \tag{53}$$

The stored energy change is:

$$\frac{dU}{dt}(t)=L\frac{d|I(t)|^2}{dt}=-\frac{\omega_0L}{Q_L}|I(\omega,\tau_0)|^2\exp\left[-\frac{\omega_0}{Q_L}(t-\tau_0)\right]=-(1+\beta_e+\beta_t)P_d(t)=-P_d(t)-P_e(t)-P_t(t) \tag{54}$$

The cavity stored energy is:

$$U(t)=\int_t^{+\infty}\frac{dU}{d\tau}(\tau)d\tau=L|I(\omega,\tau_0)|^2\exp\left[-\frac{\omega_0}{Q_L}(t-\tau_0)\right]=\frac{Q_L(1+\beta_e+\beta_t)}{\omega_0}P_d(t) \tag{55}$$

This shows that the equation (7) can be also used at this state.

The reflected power now becomes as

$$P_r(t) = -P_d(t) - P_t(t) - \frac{dU}{dt}(t) = P_e(t) = \beta_e P_d(t) \tag{56}$$

The emitted power becomes reflected power which can be measured and also has the physical meaning in this state.

According above results, the E_{acc} is:

$$\begin{aligned} E_{acc}(t) &= \sqrt{\frac{4\beta_e (R_{acc}/Q) Q_L P_{in}}{d^2(1+\beta_e+\beta_t)}}\left[1-\exp\left(-\frac{\omega_0}{2Q_L}\tau_0\right)\right]\exp\left[-\frac{\omega_0}{2Q_L}(t-\tau_0)\right] = E_{acc}(\omega,\tau_0)\exp\left[-\frac{\omega_0}{2Q_L}(t-\tau_0)\right] \\ &= \frac{\sqrt{(R_{acc}/Q)Q_0 P_d(t)}}{d} = \frac{\sqrt{(R_{acc}/Q)Q_e P_e(t)}}{d} = \frac{\sqrt{(R_{acc}/Q)Q_t P_t(t)}}{d} = \frac{\sqrt{(R_{acc}/Q)Q_e P_r(t)}}{d} \end{aligned} \tag{57}$$

From equation (53), we found

$$10\log P_r(t) = 10\log P_e(t) = 10\log P_t(t) + 10\log(\beta_e/\beta_t) = -k(t-\tau_0)+b \tag{58}$$

Here $k=10\cdot\log(e)\cdot(\omega_0/Q_L)=4.343\omega_0/Q_L$, and $b=10\log(Q_t/Q_e)$ is a constant. This equation can be used to exactly measure the loaded Q_L at pulse mode by fitting transmitted power P_t or the emitted power P_e. We can define

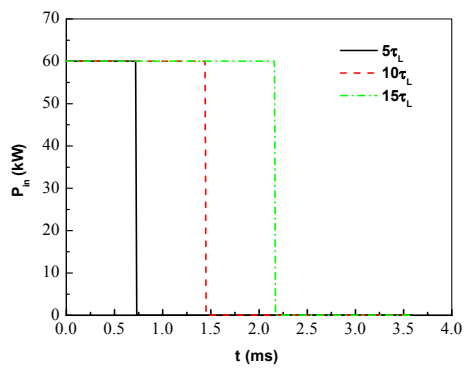
$$\frac{P_e(t)}{P_t(t)} = \frac{\beta_e}{\beta_t} = K^2 \tag{59}$$

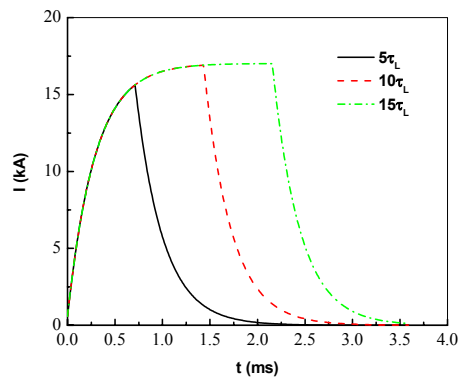
Here K is a cavity intrinsic constant, which can obtain by measuring the ratio of emitted power P_e and transmitted power P_t after RF switch off. Combining equation (59) and (38), the coupling factors are expressed as:

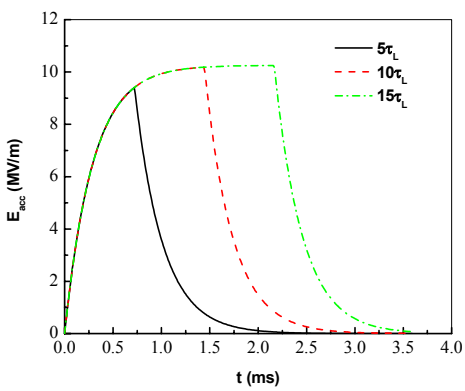
$$\begin{cases} \beta_e = \dfrac{K^2\sqrt{T(\omega_0,t)}}{2K\left[1-\exp\left(-\dfrac{\omega_0}{2Q_L}t\right)\right]-\sqrt{T(\omega_0,t)}(K^2+1)} \\ \beta_t = \dfrac{\sqrt{T(\omega_0,t)}}{2K\left[1-\exp\left(-\dfrac{\omega_0}{2Q_L}t\right)\right]-\sqrt{T(\omega_0,t)}(K^2+1)} \end{cases} \tag{60}$$

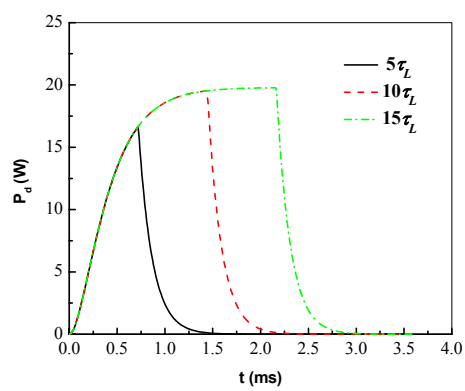
If the incident power pulse length $\tau_1 > 10Q_L/\omega_0$, then $1-\exp\left(-\frac{\omega_0}{2Q_L}\tau_1\right)\approx 1$. Then equation (60) can be approximated as:

$$\begin{cases} \beta_e = \dfrac{K^2\sqrt{T(\omega_0,\tau_1)}}{2K-\sqrt{T(\omega_0,\tau_1)}(K^2+1)} \\ \beta_t = \dfrac{\sqrt{T(\omega_0,\tau_1)}}{2K-\sqrt{T(\omega_0,\tau_1)}(K^2+1)} \end{cases} \tag{61}$$


70
60
50
40
30
20
10
0
P_in (kW)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
t (ms)
5τ_L
10τ_L
15τ_L


20
15
10
5
0
I (kA)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
t (ms)
5τ_L
10τ_L
15τ_L


12
10
8
6
4
2
0
E_acc (MV/m)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
t (ms)
5τ_L
10τ_L
15τ_L


25
20
15
10
5
0
P_d (W)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
t (ms)
5τ_L
10τ_L
15τ_L

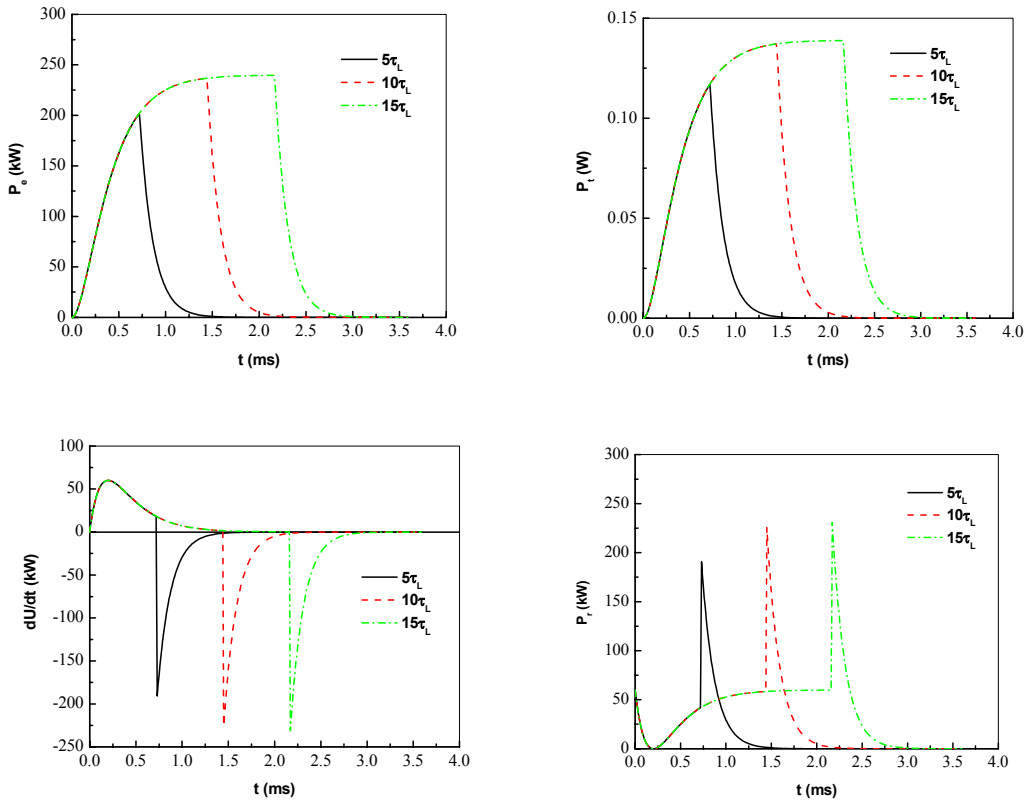


Figure 4: SNS medium beta cavity's series circuit current $I(t)$, accelerating gradient $E_{acc}(t)$, dissipated power $P_d(t)$, emitted power $P_e(t)$ (no physical meaning when P_{in}(t) is on), transmitted power $P_t(t)$, stored energy change $dU/dt(t)$ and reflected power $P_r(t)$ in different drive pulse lengths. The amplitude of the incident power P_{in} is 60 kW, the frequency is 805 MHz. We used cavity parameters in table 1. So $\tau_L=Q_L/\omega_0$=0.1435 ms.

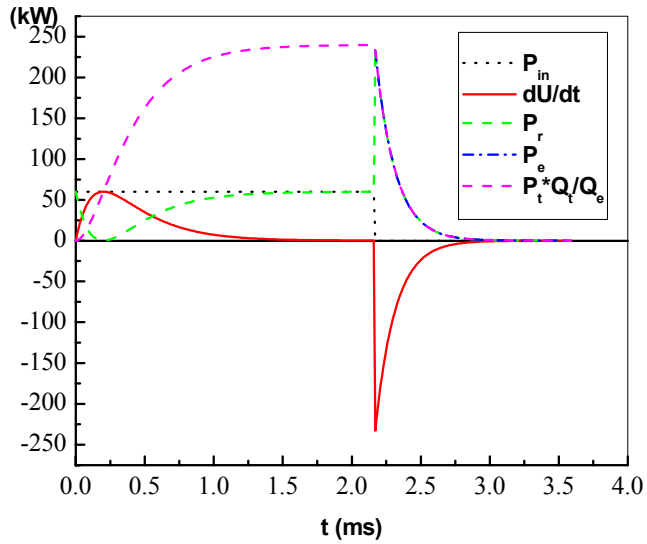


Figure 5: Combined plot from Figure 4 with pulse length of 15 τ_L.

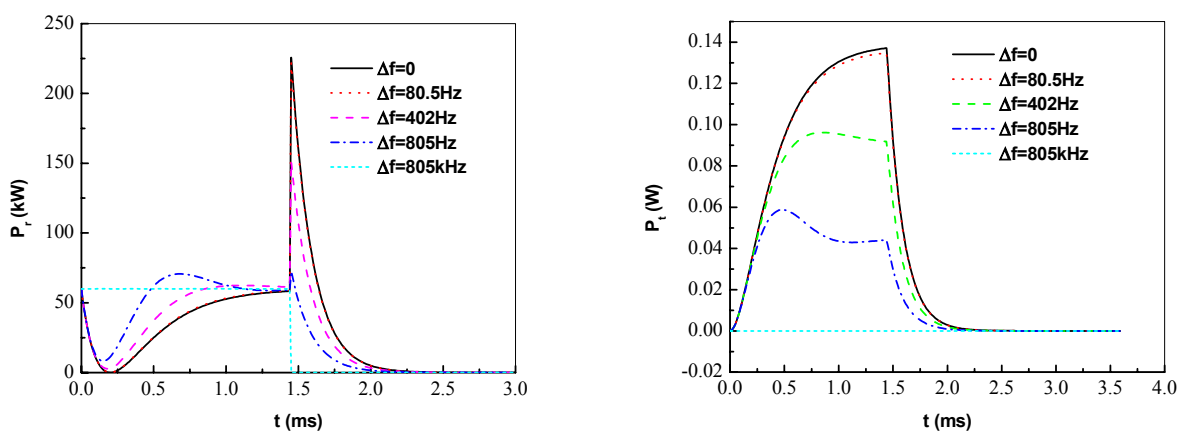


Figure 6: SNS medium beta cavity's reflected power $P_r(t)$ and transmitted power $P_t(t)$ in the drive pulse length of 10 τ_L. The amplitude of the incident power P_{in} is 60 kW, the cavity resonance frequency is 805 MHz. We used cavity parameters in table 1. So $\tau_L=Q_L/\omega_0$=0.1435 ms. $\Delta f=f-f_0$ is the frequency difference between the drive klystron and cavity's resonance frequencies.

As shown in Figure 6, the change of cavity's reflected power P_r waveforms indicates that when the cavity detuning is large enough, the all incident power will be reflected back like incident square wave. The Δf=805Hz corresponds about 65.7^o off resonance crest. The transmitted power P_t reaches maximum at cavity resonance frequency $\omega= \omega_0$. To maximize the reflected or transmitted power peak can be used to judge the optimum tuning condition of the cavity's frequency tuner or

phase.

V. Decayed Incident Power

If the incident power is square wave pulse, and its amplitude change can be ignored in the time scale of $10Q_L/\omega_0$, we can use above equations to evaluate cavity performance. If the incident power's amplitude change can not be ignored, we need to modify above equations. One such of example is the 1MW pulsed klystron installed at JLab CMTF by LANL. The total capacitance of the capacitor bank in the Pulse Forming Network (PFN) of the klystron was limited. We saw a drop during the 2ms of driving pulse.

A. RF Switch On

For a pulsed incident power which is exponentially decay with the form $P_{in}(t)=P_{in}\exp(-2\alpha t)$, the equivalent voltage becomes:

$$E(\omega,t) = E_0 \exp[(-\alpha + i\omega)t]$$

The differential equation (11) becomes:

$$L\frac{d^2 I(\omega,t)}{dt^2} + R_T \frac{dI(\omega,t)}{dt} + \frac{I(t)}{C} = (-\alpha + i\omega) n_1 E_0 \exp[(-\alpha + i\omega)t] \tag{62}$$

Using the slowly-varying amplitude approximation and the RF switch on boundary condition, the solution of the above equation is:

$$\begin{aligned} I(\omega,t) &= \frac{(-\alpha + i\omega) n_1 E_0}{(\alpha^2 L - \alpha R_T - \omega^2 L + 1/C) + i(\omega R_T - 2\alpha\omega L)} \left\{ \exp[(-\alpha + i\omega)t] - \exp\left[-\frac{R_T}{2L} t - i\left(\omega - \frac{\lambda}{2L} \right) t \right] \right\} \\ &= \frac{(-\alpha + i\omega) n_1 E_0 \exp(i\omega t)}{Q_0 R_c \omega \left[\left(\frac{\alpha^2}{\omega\omega_0} - \frac{\alpha}{\omega Q_L} - \frac{\omega}{\omega_0} + \frac{\omega_0}{\omega} \right) + i\left(\frac{1}{Q_L} - \frac{2\alpha}{\omega_0} \right) \right]} \left\{ \exp(-\alpha t) - \exp\left[-\frac{\omega_0}{2Q_L} t - i\Delta\omega t \right] \right\} \end{aligned} \tag{63}$$

Here $\Delta\omega=\omega-\omega_0$. Corresponding current module square $|I(\omega,t)|^2$ is:

$$\left|I(\omega,t)\right|^2=\frac{4\beta_e\left(\alpha^2+\omega^2\right)\exp(-2\alpha t)\left\{1+\exp\left[\left(2\alpha-\frac{\omega_0}{Q_L}\right)t\right]-2\exp\left[\left(\alpha-\frac{\omega_0}{2Q_L}\right)t\right]\cos(\Delta\omega t)\right\}}{Q_0^2R_c\omega^2\left[\left(\frac{\alpha^2}{\omega\omega_0}-\frac{\alpha}{\omega Q_L}+\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)^2+\left(\frac{1}{Q_L}-\frac{2\alpha}{\omega_0}\right)^2\right]}P_{in} \tag{64}$$

Then the cavity dissipated power $P_d(t)$ is:

$$P_d(\omega,t)=\frac{4\beta_e\left(\alpha^2+\omega^2\right)\exp(-2\alpha t)\left\{1+\exp\left[\left(2\alpha-\frac{\omega_0}{Q_L}\right)t\right]-2\exp\left[\left(\alpha-\frac{\omega_0}{2Q_L}\right)t\right]\cos(\Delta\omega t)\right\}}{Q_0^2\omega^2\left[\left(\frac{\alpha^2}{\omega\omega_0}-\frac{\alpha}{\omega Q_L}+\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)^2+\left(\frac{1}{Q_L}-\frac{2\alpha}{\omega_0}\right)^2\right]}P_{in} \tag{65}$$

and the transmitted power $P_t(t)$ is written as:

$$P_t(\omega,t)=\frac{4\beta_e\beta_t\left(\alpha^2+\omega^2\right)\exp(-2\alpha t)\left\{1+\exp\left[\left(2\alpha-\frac{\omega_0}{Q_L}\right)t\right]-2\exp\left[\left(\alpha-\frac{\omega_0}{2Q_L}\right)t\right]\cos(\Delta\omega t)\right\}}{Q_0^2\omega^2\left[\left(\frac{\alpha^2}{\omega\omega_0}-\frac{\alpha}{\omega Q_L}+\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)^2+\left(\frac{1}{Q_L}-\frac{2\alpha}{\omega_0}\right)^2\right]}P_{in}=\beta_tP_d(\omega,t) \tag{66}$$

The cavity's transmission coefficient $T(\omega_0)$ is:

$$T(\omega_0)=\frac{P_t(\omega_0,t)}{P_{in}(t)}=\frac{4\beta_e\beta_t\left(\alpha^2+\omega_0^2\right)\left\{1-\exp\left[\left(\alpha-\frac{\omega_0}{2Q_L}\right)t\right]\right\}^2}{Q_0^2\omega_0^2\left[\left(\frac{\alpha^2}{\omega_0^2}-\frac{\alpha}{\omega_0Q_L}\right)^2+\left(\frac{1}{Q_L}-\frac{2\alpha}{\omega_0}\right)^2\right]}=\frac{4\beta_e\beta_t\gamma(\omega_0)}{(1+\beta_e+\beta_t)^2}\left\{1-\exp\left[\left(\alpha-\frac{\omega_0}{2Q_L}\right)t\right]\right\}^2 \tag{67}$$

Here $\gamma(\omega_0)=\dfrac{\alpha^2+\omega_0^2}{\omega_0^2\left[\frac{\alpha^2}{\omega_0^2}\left(1-\frac{\alpha Q_L}{\omega_0}\right)^2+\left(1-\frac{2\alpha Q_L}{\omega_0}\right)^2\right]}$

The cavity's emitted power $P_e(t)$ is:

$$P_e(\omega,t)=\frac{4\beta_e^2\left(\alpha^2+\omega^2\right)\exp(-2\alpha t)\left\{1+\exp\left[\left(2\alpha-\frac{\omega_0}{Q_L}\right)t\right]-2\exp\left[\left(\alpha-\frac{\omega_0}{2Q_L}\right)t\right]\cos(\Delta\omega t)\right\}}{Q_0^2\omega^2\left[\left(\frac{\alpha^2}{\omega\omega_0}-\frac{\alpha}{\omega Q_L}+\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)^2+\left(\frac{1}{Q_L}-\frac{2\alpha}{\omega_0}\right)^2\right]}P_{in}=\beta_eP_d(\omega,t) \tag{68}$$

Again, the emitted power here has no physical meaning. Combining equations (65) (66) and (68), the cavity stored energy can be expressed as:

$$\omega U(\omega,t)=Q_0P_d(\omega,t)=Q_eP(\omega,t)_e=Q_tP_t(\omega,t)$$
$$=\frac{4\beta_e\left(\alpha^2+\omega^2\right)\exp(-2\alpha t)\left\{1+\exp\left[\left(2\alpha-\frac{\omega_0}{Q_L}\right)t\right]-2\exp\left[\left(\alpha-\frac{\omega_0}{2Q_L}\right)t\right]\cos(\Delta\omega t)\right\}}{Q_0\omega^2\left[\left(\frac{\alpha^2}{\omega\omega_0}-\frac{\alpha}{\omega Q_L}+\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)^2+\left(\frac{1}{Q_L}-\frac{2\alpha}{\omega_0}\right)^2\right]}P_{in} \tag{69}$$

Then the cavity accelerating gradient can be written as:

$$E_{acc}(\omega,t)=\sqrt{\frac{4\beta_e(R_{acc}/Q)\left(\alpha^2+\omega^2\right)P_{in}\exp(-2\alpha t)\left\{1+\exp\left[\left(2\alpha-\frac{\omega_0}{Q_L}\right)t\right]-2\exp\left[\left(\alpha-\frac{\omega_0}{2Q_L}\right)t\right]\cos(\Delta\omega t)\right\}}{d^2Q_0\frac{\omega^4}{\omega_0^2}\left[\left(\frac{\alpha^2}{\omega\omega_0}-\frac{\alpha}{\omega Q_L}+\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)^2+\left(\frac{1}{Q_L}-\frac{2\alpha}{\omega_0}\right)^2\right]}}$$
$$=\frac{\sqrt{(R_{acc}/Q)\omega_0^2Q_0P_d(t)}}{d\omega}=\frac{\sqrt{(R_{acc}/Q)\omega_0^2Q_eP_e(t)}}{d\omega}=\frac{\sqrt{(R_{acc}/Q)\omega_0^2Q_tPt(t)}}{d\omega} \tag{70}$$

The above equations show that the stored energy and accelerating gradient formula have the same form as normal square pulse's. At $\omega=\omega_0$, the accelerating gradient equation can be rewritten as:

$$E_{acc}(\omega_0,t)=\sqrt{\frac{4\beta_e(R_{acc}/Q)\left(\alpha^2+\omega_0^2\right)P_{in}}{d^2Q_0\omega_0^2\left[\left(\frac{\alpha^2}{\omega_0^2}-\frac{\alpha}{\omega_0Q_L}\right)^2+\left(\frac{1}{Q_L}-\frac{2\alpha}{\omega_0}\right)^2\right]}}\left[\exp(-\alpha t)-\exp\left(-\frac{\omega_0}{2Q_L}t\right)\right]$$
$$=\frac{\sqrt{(R_{acc}/Q)Q_0P_d(t)}}{d}=\frac{\sqrt{(R_{acc}/Q)Q_eP_e(t)}}{d}=\frac{\sqrt{(R_{acc}/Q)Q_tP_t(t)}}{d} \tag{71}$$

The cavity stored energy change dU/dt can be written as:

$$\frac{dU}{dt}(\omega,t)=\frac{2\beta_e\left(\alpha^2+\omega^2\right)\left(1+\frac{\omega_0^2}{\omega^2}\right)\left\{\begin{array}{l}\left(2\alpha+\frac{\omega_0}{Q_L}\right)\exp\left[-\left(\alpha+\frac{\omega_0}{2Q_L}\right)t\right]\cos(\Delta\omega t)-\frac{\omega_0}{Q_L}\exp\left(-\frac{\omega_0}{Q_L}t\right)\\+2\Delta\omega\exp\left[-\left(\alpha+\frac{\omega_0}{2Q_L}\right)t\right]\sin(\Delta\omega t)-2\alpha\exp(-2\alpha t)\end{array}\right\}}{\omega_0Q_0\omega^2\left[\left(\frac{\alpha^2}{\omega\omega_0}-\frac{\alpha}{\omega Q_L}+\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)^2+\left(\frac{1}{Q_L}-\frac{2\alpha}{\omega_0}\right)^2\right]}P_{in} \tag{72}$$

Certainly the cavity's stored energy is:

$$U(\omega,t)=\frac{2\beta_e\left(\alpha^2+\omega^2\right)Q_L(1+\beta_e+\beta_t)\exp(-2\alpha t)\left(1+\frac{\omega_0^2}{\omega^2}\right)\left\{1+\exp\left[\left(2\alpha-\frac{\omega_0}{Q_L}\right)t\right]-2\exp\left[\left(\alpha-\frac{\omega_0}{2Q_L}\right)t\right]\cos(\Delta\omega t)\right\}}{Q_0^2\omega_0\omega^2\left[\left(\frac{\alpha^2}{\omega\omega_0}-\frac{\alpha}{\omega Q_L}+\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)^2+\left(\frac{1}{Q_L}-\frac{2\alpha}{\omega_0}\right)^2\right]}P_{in}$$

$$=\left(1+\frac{\omega_0^2}{\omega^2}\right)\frac{Q_L(1+\beta_e+\beta_t)}{2\omega_0}P_d(\omega,t) \tag{73}$$

The cavity intrinsic quality factor is:

$$Q_0(\omega)=\frac{\omega U(\omega,t)}{P_d(\omega,t)}=\frac{\omega}{2\omega_0}\left(1+\frac{\omega_0^2}{\omega^2}\right)Q_L(1+\beta_e+\beta_t) \tag{74}$$

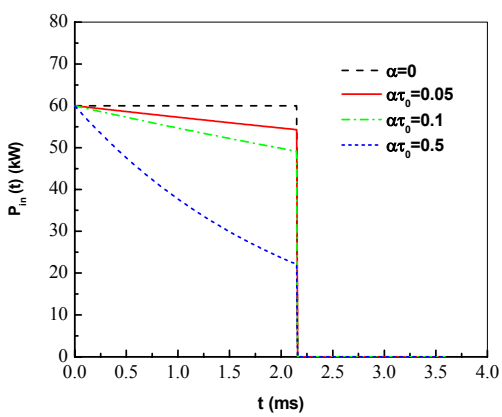
This equation shows a same format as equation (46). For $\omega=\omega_0$, the equation becomes equation (7)

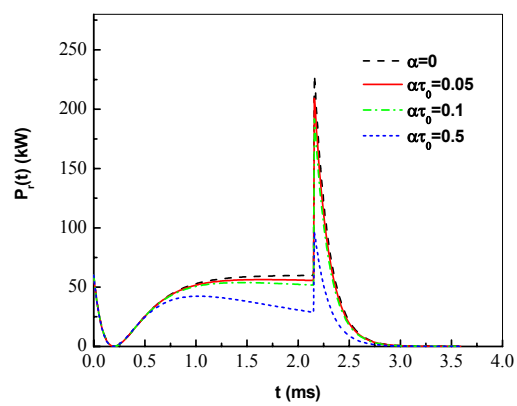
The reflected power P_r is:

$$P_r(\omega,t)=P_{in}\exp(-2at)-P_d(\omega,t)-P_t(\omega,t)-\frac{dU}{dt}(\omega,t) \tag{75}$$

B. RF Switch Off

After RF switch off, all the boundary conditions are as same as RF switch off in the normal pulse. All the equations of the decayed incident power have same form as the normal pulse's. But the physical quantities, such as $I(\tau_0)$ etc., should be replaced by the value at the pulse end.





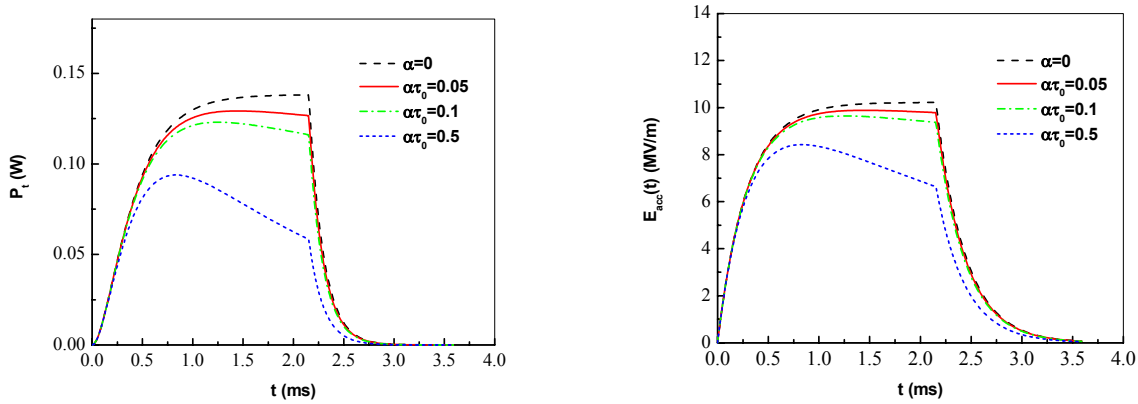


Figure 7: The SNS medium beta cavity's reflected power $P_r(t)$, transmitted power $P_t(t)$, accelerating gradient $E_{acc}(t)$ in the incident power $P_{in}(t)$ pulse length of $15\tau_0$ with decay rate of α. The amplitude of the incident power P_{in}=60kW. Drive frequency is 805Mz, and $\tau_0 = Q_L / \omega_0$=0.1435ms.

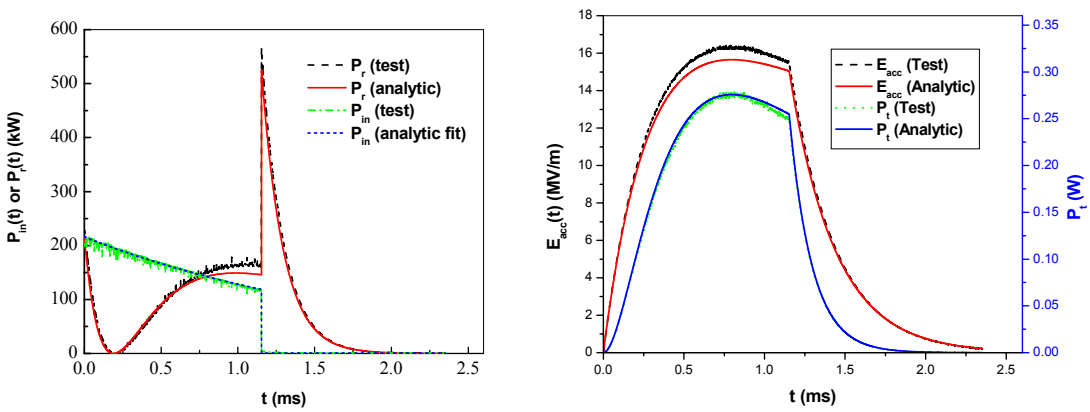


Figure 8: The SNS medium beta cavity M082 test data in JLab CMTF and comparison with the analytic fitting data in the $\alpha=0.3/\tau_0=1/3.84\ (\text{ms})^{-1}$ decay rate of incident pulse. τ_0=1.153ms. The initial amplitude of incident power is P_{in}=216kW. The E_{acc}(Test) was calculated by one-port measurement equations.

The incident power decay rate α has a strong effect on the reflected power's, transmitted power's, and cavity gradient's waveforms. As shown in Fig. 7 the decay rate α is 5%, 10% and 50% of $1/\tau_0=\omega_0/Q_L$ separately.

The incident power used to measure SNS cavities at JLab Cryomodule Test Facility (CMTF) was a decayed pulse. After fitting to the exponential decay form, we fund the decay rate is $\alpha=0.3/\tau_0=1/3.84\ (\text{ms})^{-1}$. The measurement data used in this analysis were taken on the medium beta No. 8 cryomodule cavity #2 at 9:36AM on April, 8 2004

(Waveform_040804_093645_ M08_2.txt). Substituting this α and P_{in}=216kW into equations (66), (71), (75), (65), and (72), we find that the analytic results of the reflected, transmitted power can fit the test dada very well as shown in Fig.8. The only discrepancy is on the E_{acc}, the analytical data is about 3.7% lower than measured data at top. The test data was calculated by one-port measurement equations [7].

Conclusion

The two-port RF cavity's measurement equations based a series equivalent circuit can accurately measure the cavity's parameters without beam load, such as the cavity in a vertical Dewar or in a horizontal cryomodule. When coupling strength β_e is closed to 1, these equations can be even used to measure the cavity's intrinsic quality factor Q_0. Substituting β_t=0 into the two-port cavity equations, the two-port equations be simplified into one-port cavity equations [1-4].

In the pulsed mode operation, the cavity stored energy change dU/dt and FPC coupling are the cause of the reflected power P_r and transmitted power P_t. Emitted power P_e only has the physical meaning when the drive power is switched off. It becomes reflected power P_r and is external Q ratio of FP to FPC times of transmitted power P_t. In practice, the incident power P_{in} is not always square wave pulse. To measure cavity's parameters more accurately, like loaded quality factor Q_L, accelerating gradient E_{acc}, the measurement equations need to be modified.

LRC series equivalent circuit is a good physical model to measure the electrical characteristics of superconducting RF cavity.

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References

1. H. Padamsee, J. Knobloch, and T. Hays, **RF Superconductivity for Accelerators**, (Wiley, New York, 1998)
2. J. Weaver, **Formulas for Measuring Q_0, β, E_{max} and B_{max} in a Superconducting Cavity**, SLAC-TN-21, July 1968.

3. G. A. Loew, **Charging and Discharging Superconducting Cavities**, SLAC-TN-68-25, November 1968.

4. L. Merminga, **RF Cavity Equations: Steady-State**, JLab-TN-95-019, July 18, 1995.

5. C. Montgomery, R. Dicke and E. Purcell, **Principles of Microwave Circuits**, (Dover Publications, INC., New York, 1965).

6. M. Sargent III, M. Scully and W. Lamb, **Laser Physics**, (Addison-Wesley Publishing Company, CA. 1974).

7. C. Reece, T Powers, P. Kushnick, **An Automated RF Control and Data Acquisition System for Testing Superconducting RF Cavities**, CEBAF-TN-91-031 **,** also known as CEBAF-SRF-91-05-03-EXA.