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Curvature-Induced Bunch Self-Interaction for an Energy-Chirped Bunch in Magnetic Bends

Rui Li

Thomas Jefferson National Accelerator Facility,

12000 Jefferson Ave.,

Newport News, VA 23606, USA

Abstract

The curvature-induced bunch collective interaction in magnetic bends can be studied using effective forces in the canonical formulation of the coherent synchrotron radiation (CSR) effect. In this paper, for an electron distribution moving ultrarelativistically in a bending system, the dynamics of the particles in the electron distribution is derived from the Hamiltonian of the particles in terms of the bunch internal coordinates. The consequent Vlasov equation manifests explicitly how the phase space distribution is perturbed by the effective CSR forces. In particular, we study the impact of an initial linear energy chirp of the bunch on the behavior of the effective longitudinal CSR force, which arises due to the modification of the retardation relation as a result of the energy-chirping-induced longitudinal-horizontal correlation of the bunch distribution (bunch tilt) in dispersive regions.

1. INTRODUCTION

In linear accelerators or linac drivers for free-electron lasers, often a linear energy chirp, or a linear correlation between a particle's energy and its longitudinal position in a bunch, is imposed on a bunch by accelerating the bunch off-crest in an RF cavity. Transporting such an energy-chirped bunch through a bending system allows the manipulation of the bunch length via the pathlength-energy correlation in dispersive regions. For example, a high peak current of an electron beam is often achieved by compressing the high charge, properly energy-chirped electron bunches using a magnetic bunch compression chicane. When such an electron bunch with linear energy chirp (δ-z correlation) going through a dispersive region (x-δ correlation), such as that in a magnetic chicane, horizontal-longitudinal (x-z) correlation, or bunch tilt, is introduced to the bunch distribution. Here we use the LCLS BC2 chicane described in Ref. [2] to illustrate this x-z correlation. The chicane consists of four dipole magnets of length 0.4 m with bending radius of 12.2 m. The two center dipoles are 0.5 m apart, and the two side drifts are 10 m each. For an initial δ-z correlation $u = -40$ m^{-1}, Fig. 1 and Fig. 2 respectively depict the compression factor and the x-z slope of the bunch, $\xi(s)$, along the beamline. Note that at the end of the 3rd bend when $s = 11.3$ m, $\xi(s)$ reaches its maximum: $\xi(s) = 118$.

The coherent synchrotron radiation (CSR) effect on microbunching instability in a bunch compression chicane has been extensively studied both analytically [1–3] and numerically [4]. As an approximation, these studies are based on the longitudinal CSR wakefield [5–7] obtained for a one-dimensional bunch, which is the projection of an actual tilted bunch onto the designed circular orbit. The effective CSR forces were also analyzed earlier [8] for a nontilted bunch. Compared to the case of a nontilted or projected bunch, the x-z correlation of the bunch distribution modifies the geometry of particle interaction with respect to the direction of particle motion, which consequently modifies the retardation solution and the CSR interaction force. The numerical results of the full two-dimensional effect, which account for the x-z correlation of the bunch distribution in configuration space, were first presented by Dohlus [9]. In this study we focus on the analysis of this bunch tilt effect on the CSR interaction.

Since the variation of bunch tilt depends closely on the beam phase space transport along the beamline of interest, to study its effect on the curvature-induced collective forces, we need to start from a more general approach, i.e., to formulate the bunch collective interaction on a curved orbit as a self-consistent dynamical system for an arbitary initial phase space distribution. This is done by first developing the equations of motion in Sec. 2 based on the Hamiltonian of the particles in the bunch distribution, and then constructing the equation for the evolution of the phase space distribution in Sec. 3, where the role of the effective collective forces on the perturbation of the phase space distribution is explicitly shown. We then focus on the study of the CSR interaction for a 2D energy-chirped bunch on a circular orbit with the zeroth order approximation, which includes the formulation of the retarded potentials (Sec. 4), the solutions of the retardation relation (Sec. 5), and the analysis of the effective longitudinal CSR force (Sec. 6). The analytical results obtained in this paper are in good agreement with Dohlus' numerical results for a moderately tilted bunch.

It is well-known that when a charge distribution moves ultrarelativistically along a curved orbit, the particles experience the "centrifugal space-charge force" $F^{\rm CSCF}$ in the radial direction [10], as a result of the non-perfect cancellation between the electric and magnetic fields in the Lorentz force. Meanwhile, the particles with radial offset from the design orbit also experience the "noninertial space-charge force" $F^{\rm NSCF}$ in the longitudinal direction [11]. The singular contribution of nearby particle interaction to these forces could cause difficulties for one to analyze the curvature-induce collective interactions for a tilted bunch. In this paper, this difficulty is eased by using the canonical formulation developed in Ref. [12], which exhibits explicitly the anti-correlation (or cancellation) of the effects of $F^{\rm CSCF}$ and of the potential change (which is partly contributed by $F^{\rm NSCF}$) on the bunch transverse dynamics and microbunching process. Consequently, the effective CSR forces, which are the net residual of this cancellation, are recongnized as the source of perturbation to the phase space distribution. Compared to $F^{\rm CSCF}$ and $F^{\rm NSCF}$, the effective forces are usually dominated by non-singular contributions of particle interactions.

Even though in this paper our focus is limited to the "steady-state" longitudinal effective force for a tilted bunch, the basic equations developed in Sec. 2 and Sec. 3 provide a framework for further self-consistent analysis or simulation of the CSR effect on the dynamics of an energy-chirped bunch in bends. Moreover, for a tilted bunch, the approach used in Sec. 4-6 can be straightforwardly extended to the "steady-state" transverse effective CSR force, as well as to the effective forces in the transient regime involving entrance to and exit from a circular orbit.

2. FORMULATION OF THE CSR PROBLEM AS A DYNAMICAL SYSTEM

Our goal in this section is to formulate particle dynamics for an electron distribution moving ultrarelativistically on a curved design orbit under collective interaction in free space (impedance effects due to boundary condition are not included). With a brief discussion of the dynamics in the Cartesian coordinates in the laboratory frame, we review the equations of motion based on the canonical formulation [12] in the Frenet-Serret coordinates along the design orbit, using pathlength s as the independent variable. Then by writing both the equations of motion and the effective forces in terms of our chosen beam phase space parameters, we formulate the problem as a dynamical system (see Eq. (22)). This provides the foundation for setting up the equation for the evolution of the phase space distribution (see Sec. 3), and allows us to analyze the curvature-induced effective forces for an energy-chirped bunch in magnetic bends (Sec. 4-6).

2.1. Dynamics in a Cartesian Coordinate System

Consider an electron bunch moving relativistically in an external electromagnetic field and undergoing collective interaction. For a fixed Cartesian coordinate system, the position vector is $\mathbf{r} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$. The dynamics of a charged particle (with respect to t) in the bunch can be studied using the Hamiltonian

$$H = \sqrt{(cP_x - eA_x)^2 + (cP_y - eA_y)^2 + (cP_z - eA_z)^2 + m^2c^4} + e\Phi. \tag{1}$$

Here $\mathbf{P}$ is the canonical momentum conjugate to $\mathbf{r}$, and

$$\Phi = \Phi^{\rm ext} + \Phi^{\rm col}, \qquad \mathbf{A} = \mathbf{A}^{\rm ext} + \mathbf{A}^{\rm col} \tag{2}$$

are the scalar and vector potentials on the charged particle, with $(\Phi^{\rm ext}, \mathbf{A}^{\rm ext})$ the potentials related to the external design fields, and $(\Phi^{\rm col}, \mathbf{A}^{\rm col})$ the potentials due to collective electromagnetic interactions among particles in the charge distribution. Let $\mathbf{r}_0(t;\zeta)$ denote the trajectory of a source particle, with $\zeta = (\mathbf{r}(0), \dot{\mathbf{r}}(0))$ the source particle's initial phase space parameter, and let $f_0(\zeta)$ be the initial phase space distribution of the electron bunch. From the retarded potential generated by a single particle [13],

$$A^\mu(x) = 2e\int d\tau V^\mu(\tau)\theta[x_0 - r_0(\tau)]\delta\{[x - r(\tau)]^2\} \tag{3}$$

for $V^\mu(\tau) = (\gamma c, \gamma\dot{\mathbf{r}})$, one can obtain the collective potentials (in Lorentz gauge) on a test particle generated by all the source particles in the electron distribution,

$$\begin{cases} \Phi^{\rm col}(\mathbf{r},t) &= 2ec\int d\zeta f_0(\zeta)\int dt_r\,\theta(t-t_r)\,\delta\left[c^2(t-t_r)^2 - |\mathbf{r}-\mathbf{r_0}(t_r,\zeta)|^2\right] \\ \mathbf{A}^{\rm col}(\mathbf{r},t) &= 2e\int d\zeta f_0(\zeta)\int dt_r\,\dot{\mathbf{r}}(t_r)\,\theta(t-t_r)\,\delta\left[c^2(t-t_r)^2 - |\mathbf{r}-\mathbf{r_0}(t_r,\zeta)|^2\right], \end{cases} \tag{4}$$

where $\theta(x)$ is the Heaviside step function, and $\delta(x)$ is the Dirac-delta function.

2.2. Dynamics in the Bunch Internal Coordinate System

We now let the pathlength s along the design orbit be the independent variable, and summarize the equations of motion in the Frenet-Serret coordinates [12]. Here the position vector of a particle is

$$\mathbf{r}(x,y,s) = x\,\mathbf{e}_x(s) + y\,\mathbf{e}_y(s) + \mathbf{e}_s(s). \tag{5}$$

Let H be the canonical energy conjugate to t, and $\mathcal{P}_x$ and $\mathcal{P}_y$ be the transverse canonical momentums conjugate to x and y. The Hamiltonian for an electron in the bunch is then

$$\mathcal{H}(x,\mathcal{P}_x,y,\mathcal{P}_y,t,-H;s) = -(1+\kappa_0 x) \\ \times\left[\frac{e}{c}A_s^{\rm ext} + \frac{e}{c}A_s^{\rm (F)} + \sqrt{\left(\frac{H-e\Phi^{\rm (F)}}{c}\right)^2 - \left(\mathcal{P}_x - \frac{e}{c}A_x^{\rm (F)}\right)^2 - \left(\mathcal{P}_y - \frac{e}{c}A_y^{\rm (F)}\right)^2 - m^2c^2}\,\right], \tag{6}$$

where $\kappa_0(s) = 1/R_0(s)$ is the curvature of the design orbit at s, and $\Phi^{\rm (F)}$ and $\mathbf{A}^{\rm (F)}$ are the collective interaction potentials in Eq. (4) converted to the Frenet-Serrent coordinates using Eq. (5):

$$\Phi^{\rm (F)}(x,y,s,t) = \Phi^{\rm col}(\mathbf{r}(x,y,s),t), \qquad \mathbf{A}^{\rm (F)}(x,y,s,t) = \mathbf{A}^{\rm col}(\mathbf{r}(x,y,s),t). \tag{7}$$

Here the Frenet components of $\mathbf{A}^{\rm (F)}(x,y,z,s)$ are

$$A_x^{\rm (F)}(x,y,t,s) = \mathbf{A}^{\rm (F)}\cdot\mathbf{e}_x(s), \quad A_y^{\rm (F)}(x,y,t,s) = \mathbf{A}^{\rm (F)}\cdot\mathbf{e}_y(s), \quad A_s^{\rm (F)}(x,y,z,s) = \mathbf{A}^{\rm (F)}\cdot\mathbf{e}_s(s). \tag{8}$$

Normalizing the Hamiltonian in Eq. (6) by $\gamma_0 mc = E_0/c$, with E_0 being the design beam energy, one gets

$$\tilde{\mathcal{H}}(x,\tilde{\mathcal{P}}_x,y,\tilde{\mathcal{P}}_y,ct,-\tilde{H};s) = \\ -(1+\kappa_0 x)\left[\tilde{A}_s^{\rm ext} + \tilde{A}_s^{\rm (F)} + \sqrt{(\tilde{H}-\tilde{\Phi}^{\rm (F)})^2 - (\tilde{\mathcal{P}}_x - \tilde{A}_x^{\rm (F)})^2 - (\tilde{\mathcal{P}}_y - \tilde{A}_y^{\rm (F)})^2 - \gamma_0^{-2}}\,\right] \tag{9}$$

with

$$\left\{\tilde{\mathcal{H}}, \tilde{H}, \tilde{\mathcal{P}}_x, \tilde{\mathcal{P}}_y\right\} = \frac{1}{\gamma_0 mc}\left\{\mathcal{H}, \frac{H}{c}, \mathcal{P}_x, \mathcal{P}_y\right\} \tag{10}$$

and

$$\left\{\tilde{A}_s^{\rm ext}, \tilde{A}^{\rm (F)}, \tilde{\Phi}^{\rm (F)}\right\} = \frac{e}{\gamma_0 mc^2}\left\{A_s^{\rm ext}, A^{\rm (F)}, \Phi^{\rm (F)}\right\} \tag{11}$$

for $A^{\rm (F)}$ being one of $A^{\rm (F)}_{x,y,s}$.

A canonical transformation from $(ct, -\tilde{H})$ to (z, δ_H) can be performed using the following generating function [14]

$$F_3(-\tilde{H}, z, s) = (s-z)\sqrt{\tilde{H}^2 - \gamma_0^{-2}} + z, \tag{12}$$

which gives

$$\begin{cases} \delta_H = -\dfrac{\partial F_3}{\partial z} = \sqrt{\tilde{H}^2 - \gamma_0^{-2}} - 1 \\ ct = \dfrac{\partial F_3}{\partial \tilde{H}} = \dfrac{s-z}{\beta_H} \\ \tilde{\mathcal{K}} = \tilde{\mathcal{H}} + \dfrac{\partial F_3}{\partial s} \end{cases} \tag{13}$$

for $\beta_H = \sqrt{1 - (\gamma_0 \tilde{H})^{-1}}$. Let us define the normalized potentials with variables changed according to Eq. (13) as

$$\left\{\tilde{\Phi}, \tilde{A}_{x,y,s}, \tilde{\mathcal{A}}_s^{\rm ext}\right\}(x, y, z, \delta_H, s) = \left\{\tilde{\Phi}^{\rm (F)}, \tilde{A}^{\rm (F)}_{x,y,s}, (1+\kappa_0 x)\tilde{A}_s^{\rm ext}\right\}\left(x, y, s, t = \frac{s-z}{\beta_H(\delta_H)c}\right), \tag{14}$$

where $\tilde{\mathcal{A}}_s^{\rm ext} = (1+\kappa_0 x)A_s^{\rm ext}$ is the canonical external longitudinal vector potential. Applying Eq. (9) to the new Hamiltonian $\tilde{\mathcal{K}}$ in Eq. (13), and expanding the square root in Eq. (9) to the 2nd order of the small quantities

$$\delta_H, \tilde{\Phi}, (\tilde{\mathcal{P}}_x - \tilde{A}_x), (\tilde{\mathcal{P}}_y - \tilde{A}_y), 1/\gamma_0, \tag{15}$$

we get the approximation of $\tilde{\mathcal{K}}$

$$\begin{aligned} &\tilde{\mathcal{K}}(x, \tilde{\mathcal{P}}_x, y, \tilde{\mathcal{P}}_y, z, \delta_H; s) \simeq \\ &\quad -\tilde{\mathcal{A}}_s^{\rm ext} - \kappa_0 x(1+\delta_H) - (1+\kappa_0 x)\left[(\tilde{A}_s - \tilde{\Phi}) - \frac{(\tilde{\mathcal{P}}_x - \tilde{A}_x)^2 + (\tilde{\mathcal{P}}_y - \tilde{A}_y)^2}{2}\right]. \end{aligned} \tag{16}$$

Since to the second order, for $\beta_0 = \sqrt{1 - \gamma_0^{-2}}$, one has

$$\beta_H(\delta_H) = \left[1 - \frac{1}{\gamma_0^2(1+\delta_H)^2 + 1}\right]^{1/2} \simeq \beta_0, \tag{17}$$

the potentials in Eq. (16) can be approximated from Eq. (14)

$$\left\{\tilde{\Phi}, \tilde{A}_{x,y,s}, \tilde{\mathcal{A}}_s^{\rm ext}\right\}(x,y,z,s) \simeq \left\{\tilde{\Phi}^{(\rm F)}, \tilde{A}^{(\rm F)}_{x,y,s}, (1+\kappa_0 x)\tilde{A}_s^{\rm ext}\right\}\left(x,y,z,t=\frac{s-z}{\beta_0 c}\right). \qquad (18)$$

The Hamilton's equation for $(x, \tilde{\mathcal{P}}_x, y, \tilde{\mathcal{P}}_y, z, \delta_H)$ can then be obtained after applying the potentials in Eq. (18) to $\tilde{\mathcal{K}}$ in Eq. (16):

$$\begin{cases} \dfrac{dx}{ds} = (1+\kappa_0 x)\,(\tilde{\mathcal{P}}_x - \tilde{A}_x), \\ \dfrac{d\mathcal{P}_x}{ds} = \left[\dfrac{\partial \tilde{\mathcal{A}}_s^{\rm ext}}{\partial x} + \kappa_0(1+\delta_H)\right] + \kappa_0\left[(\tilde{A}_s - \beta_0\tilde{\Phi}) - \dfrac{(\tilde{\mathcal{P}}_x - \tilde{A}_x)^2 + (\tilde{\mathcal{P}}_y - \tilde{A}_y)^2}{2}\right] \\ \qquad\qquad + (1+\kappa_0 x)\left[\dfrac{\partial(\tilde{A}_s - \beta_0\tilde{\Phi})}{\partial x} + (\tilde{\mathcal{P}}_x - \tilde{A}_x)\dfrac{\partial \tilde{A}_x}{\partial x} + (\tilde{\mathcal{P}}_y - \tilde{A}_y)\dfrac{\partial \tilde{A}_y}{\partial x}\right] \\ \dfrac{dy}{ds} = (1+\kappa_0 x)\,(\tilde{\mathcal{P}}_y - \tilde{A}_y), \\ \dfrac{d\mathcal{P}_y}{ds} = \dfrac{\partial \tilde{\mathcal{A}}_s^{\rm ext}}{\partial y} + (1+\kappa_0 x)\left[\dfrac{\partial(\tilde{A}_s - \beta_0\tilde{\Phi})}{\partial y} + (\tilde{\mathcal{P}}_x - \tilde{A}_x)\dfrac{\partial \tilde{A}_x}{\partial y} + (\tilde{\mathcal{P}}_y - \tilde{A}_y)\dfrac{\partial \tilde{A}_y}{\partial y}\right] \\ \dfrac{dz}{ds} = -\kappa_0 x \\ \dfrac{dH}{ds} = \dfrac{\partial \tilde{\mathcal{A}}_s^{\rm ext}}{\partial z} + (1+\kappa_0 x)\left[\dfrac{\partial(\tilde{\mathcal{A}}_s - \beta_0\tilde{\Phi})}{\partial z} + (\tilde{\mathcal{P}}_x - \tilde{A}_x)\dfrac{\partial \tilde{A}_x}{\partial z} + (\tilde{\mathcal{P}}_y - \tilde{A}_y)\dfrac{\partial \tilde{A}_y}{\partial z}\right] \end{cases} . \qquad (19)$$

Let us use the external canonical vector potential [15]

$$\tilde{\mathcal{A}}_s^{\rm ext}(x,y,s) = -\left[\kappa_0(s)x + \left(\kappa_0^2(s) - \kappa_1(s)\right)\frac{x^2}{2} + \kappa_1(s)\frac{y^2}{2}\right] + \cdots \qquad (20)$$

to represent the external dipole and quadruple magnetic fields

$$\mathbf{B}^{\rm ext} = \frac{E_0}{e}\left[\,\kappa_1(s) y\,\mathbf{e}_x(s) - \,(\kappa_0(s) + \kappa_1(s)x)\ \mathbf{e}_y(s)\right] + \cdots \quad . \qquad (21)$$

We can then obtain the equations of motion for the noncanonical dynamical variables $(x, x', y, y', z, \delta_H)$ by rewriting Eq. (19). Since we are interested in the perturbation of the effective CSR force on the linear optics, by keeping only the linear terms for the design optics, we get

$$\frac{dX}{ds} = Y(X,s) \equiv M(s)X + F^{[f]}(X,s) \qquad (22)$$

for

$$X = \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta_H \end{pmatrix}, \quad M(s) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -k_x^2(s) & 0 & 0 & 0 & 0 & \kappa_0(s) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -k_y^2(s) & 0 & 0 & 0 \\ -\kappa_0(s) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad F^{[f]}(X,s) = \begin{pmatrix} 0 \\ \tilde{F}_x \\ 0 \\ \tilde{F}_y \\ 0 \\ \tilde{F}_H \end{pmatrix}. \tag{23}$$

Here $M(s)X$ in Eq. (22) reflects the nominal linear optics, with the horizontal and vertical focusing strengths $k_x(s)$ and $k_y(x)$ of external magnetic fields satisfying $k_x^2(s) = \kappa_1(s) - \kappa_0^2(s)$ and $k_y^2(s) = -\kappa_1(s)$. The term $F^{[f]}(X,s)$ contains the normalized effective CSR forces as expressed in terms of potentials:

$$\begin{cases} \tilde{F}_x(X,s) \;\simeq\; (1+\kappa_0 x)\left[\kappa_0(\tilde{A}_s - \beta_0\tilde{\Phi}) + (1+\kappa_0 x)\dfrac{\partial(\tilde{A}_s - \beta_0\tilde{\Phi})}{\partial x}\right. \\ \qquad\qquad \left. +y'\left(\dfrac{\partial \tilde{A}_y}{\partial x} - \dfrac{\partial \tilde{A}_x}{\partial y}\right) - \left(\dfrac{\partial \tilde{A}_x}{\partial s} - \kappa_0 x\dfrac{\partial \tilde{A}_x}{\partial z}\right)\right] \\ \tilde{F}_y(X,s) \;\simeq\; (1+\kappa_0 x)\left[(1+\kappa_0 x)\dfrac{\partial(\tilde{A}_s - \beta_0\tilde{\Phi})}{\partial y}\right. \\ \qquad\qquad \left. +x'\left(\dfrac{\partial \tilde{A}_x}{\partial y} - \dfrac{\partial \tilde{A}_y}{\partial x}\right) - \left(\dfrac{\partial \tilde{A}_y}{\partial s} - \kappa_0 x\dfrac{\partial \tilde{A}_y}{\partial z}\right)\right] \\ \tilde{F}_H(X,s) \;\simeq\; (1+\kappa_0 x)\dfrac{\partial(\tilde{A}_s - \beta_0\tilde{\Phi})}{\partial z} + x'\dfrac{\partial \tilde{A}_x}{\partial z} + y'\dfrac{\partial \tilde{A}_y}{\partial z} \end{cases}. \tag{24}$$

Note that for $U = \tilde{A}_x$ or $\tilde{A}_y$, the expansion

$$\frac{dU}{ds} = x'\frac{\partial U}{\partial x} + y'\frac{\partial U}{\partial y} + z'\frac{\partial U}{\partial z} + \frac{\partial U}{\partial s} \tag{25}$$

is used in deriving Eqs. (22)-(24). In Eq. (24) higher-order terms were included so one can compare the orders of magnitude between the dominant terms and the negligible terms.

Let $f(X,s)$ denote the phase space distribution function of the bunch. The superscript $[f]$ of $F^{[f]}$ in Eq. (22) shows the functional dependence of the effective CSR forces on $f(X,s)$ through the dependence of the scalar and vector potentials on the charge phase space distribution. Let N be the total number of electrons in the bunch; then for $dX = dx\,dx'\,dy\,dy'\,dz\,d\delta_H$,

$$\int f(X,s)dX = N. \tag{26}$$

Instead of $f_0(\zeta)$ with $\zeta = (\mathbf{r}(0), \dot{\mathbf{r}}(0))$ in Eq. (4), here we use the initial bunch phase space distribution $f_0(\zeta)$ with $\zeta = X|_{s=0}$. The particle number conservation then requires $f_0(\zeta)d\zeta = f(X,s)dX$ for an infinitesimal initial phase space volume $d\zeta$. This enables us to write the the collective interaction potentials in Eq. (4) on a test particle in terms of $f(X,s)$. Using Eqs. (4), (7) and (18), and changing variable from t_r to s_r via $t_r = [s_r - z(s_r,\zeta)]/\beta_0 c$, with

$$\frac{dr^\mu(s_r)}{ds_r} = \left[\frac{1+\kappa_0(s_r)x_r}{\beta_0}, [1+\kappa_0(s_r)x_r]\,\mathbf{e}_s(s_r) + x'_r\,\mathbf{e}_x(s_r) + y'_r\mathbf{e}_y(s_r)\right], \tag{27}$$

we have

$$\left\{\begin{array}{rcl}\tilde{\Phi}(x,y,z,s) &=& \dfrac{2\beta_0 r_e}{\gamma_0}\displaystyle\int ds_r \int dX_r f(X_r,s_r)\,[1+\kappa_0(s_r)x_r] \\ \multicolumn{3}{l}{\times\theta[(s-s_r)-(z-z_r)]\,\delta\{[(s-s_r)-(z-z_r)]^2 - \beta_0^2[\mathbf{r}(x,y,s)-\mathbf{r}(x_r,y_r,s_r)]^2\}} \\ \\ \tilde{\mathbf{A}}(x,y,z,s) &=& \dfrac{2\beta_0^2 r_e}{\gamma_0}\displaystyle\int ds_r \int dX_r f(X_r,s_r)\,\{(1+\kappa_0(s_r)x_r)\,\mathbf{e}_s(s_r) + x'_r\,\mathbf{e}_x(s_r) + y'_r\mathbf{e}_y(s_r)\} \\ \multicolumn{3}{l}{\times\theta[(s-s_r)-(z-z_r)]\,\delta\{[(s-s_r)-(z-z_r)]^2 - \beta_0^2[\mathbf{r}(x,y,s)-\mathbf{r}(x_r,y_r,s_r)]^2\},}\end{array}\right. \tag{28}$$

with r_e the electron radius, $X_r = (x_r, x'_r, y_r, y'_r, z_r, \delta_{Hr})$ representing the source particle's phase space variables at the retarded pathlength s_r, and $\mathbf{r}(x,y,s)$ and $\mathbf{r}(x_r,y_r,s_r)$ the position vectors (see Eq. (5)) for the test and source particles respectively. In Eq. (28), the Dirac-delta function and the Heaviside step function together imply that the source particle's retarded longitudinal position is determined by

$$z_r = z_r(x,y,z,s;x_r,y_r,s_r) \equiv z - s + s_r + \beta_0|\mathbf{r}(x,y,s) - \mathbf{r}(x_r,y_r,s_r)|, \tag{29}$$

which imposes the retardation relation

$$t_r = t - \frac{|\mathbf{r}(x,y,s) - \mathbf{r}(x_r,y_r,s_r)|}{c} \tag{30}$$

on the interaction between particles.

Another familiar expression for the potentials, with $X_r^\perp = (x_r, x_r', y_r, y_r')$, is

$$\left\{\begin{array}{rcl}\tilde{\Phi}(x,y,z,s) &=& \dfrac{r_e}{\gamma_0}\displaystyle\int ds_r \int dX_r^\perp d\delta_{Hr}\,[1+\kappa_0(s_r)x_r] \\ && \times\dfrac{f\left[X_r^\perp, z_r(x,y,z,s;x_r,y_r,s_r),\delta_{Hr},s_r\right]}{|\mathbf{r}(x,y,s)-\mathbf{r}(x_r,y_r,s_r)|} \\ \tilde{\mathbf{A}}(x,y,z,s) &=& \dfrac{\beta_0 r_e}{\gamma_0}\displaystyle\int ds_r \int dX_r^\perp d\delta_{Hr}\,\left[(1+\kappa_0(s_r)x_r)\,\mathbf{e}_s(s_r)+x_r'\,\mathbf{e}_x(s_r)+y_r'\mathbf{e}_y(s_r)\right] \\ && \times\dfrac{f\left[X_r^\perp, z_r(x,y,z,s;x_r,y_r,s_r),\delta_{Hr},s_r\right]}{|\mathbf{r}(x,y,s)-\mathbf{r}(x_r,y_r,s_r)|}\end{array}\right. . \tag{31}$$

Here the potentials depend on z only through $z_r(x,y,z,s,x_r,y_r,s_r)$, which makes the study of the longitudinal effective forces easier in Sec. 6.

Note that for a particle with kinetic energy $E=(1+\delta_E)E_0$, using $H=E+e\Phi$ and Eq. (13), one has

$$\delta_H = \delta_E + \tilde{\Phi} - \frac{1}{2\gamma_0^2}. \tag{32}$$

The reasons for choosing the special combination $X=(x,x',y,y',z,\delta_H)^T$ as the dynamical variables in this paper are: (1) for vanishing effective CSR force $F^{[f]}$, we restore the nominal design optics from Eq. (22); (2) the phase space distribution $f(X,s)$ can be directly related to many of the quantities measurable in laboratories (see Appendix B.2); and (3) this set of phase space parameters allows the cancellation effect to be explicitly manifested in the equations of motion, as can be seen in the first term of $\tilde{F}_x$ in Eq. (24) (see Ref. [12, 16] for more details).

3. THE EVOLUTION OF PHASE SPACE DISTRIBUTION

In this section, the evolution equation for the charge distribution in our chosen phase space is set up based on the dynamical equations formulated in Sec. 2, which is further transformed into an integral equation for an initially energy-chirped bunch. The relations of the phase space distribution $f(X,s)$ with the physical quantities observed in laboratories are also listed.

3.1. Setting up the Evolution Equation

Since we choose $(x, x', y, y', z, \delta_H)$ as the phase space variables, which are different from $(x, x', y, y', z, \delta_E)$ as usually used in the Vlasov equation for many other accelerator physics problems, we need to set up the phase space evolution equation for the particular dynamical system described by Eq. (22).

From Eq. (22), we have the phase space variables at s and at $s + \Delta s$ related to the first order of Δs by

$$X_1 \equiv X(s + \Delta s) = X(s) + Y(X, s)\Delta s. \tag{33}$$

Note that just as in the case of the Maxwell-Vlasov equation, here the phase space volume is conserved to the first order of Δs, namely,

$$dX_1 = [\, 1 + \Gamma^{[f]}(X, s)\, \Delta s\,]\, dX = dX, \tag{34}$$

where from Eq. (22)

$$\Gamma^{[f]}(X, s) \equiv \sum_{i=1}^{6} \frac{\partial Y_i(X, s)}{\partial X_i} = 0. \tag{35}$$

Using Eq. (34), and the conservation of the number of charged particles in an infinitesimal phase space volume

$$dN = f(X, s)dX = f(X_1, s + \Delta s)dX_1, \tag{36}$$

one gets the Vlasov equation for the dynamics given by Eq. (22)

$$\frac{\partial f}{\partial s} + \sum_{i=1}^{6} Y_i(X, s)\, \frac{\partial f}{\partial X_i} = 0. \tag{37}$$

3.2. Integral Equation for an Energy-Chirped Bunch

Following the analysis in Ref. [2, 3], we now construct the integral equation using our phase space evolution equation in Eq. (37). This integral equation shows how the particle distribution in the normalized phase space, which is invariant under design linear optics, is perturbed by the effective CSR forces.

For a bunch with an initial energy chirp $\delta_E = uz$ ($u < 0$ for a bunch under compression) imposed by an upstream RF cavity, let $\rho(X_0, s)$ be the particle phase space distribution for the normalized phase space variables X_0, as introduced in Appendix A. From Eqs. (A7) we have

$$f(X, s) = \rho[X_0 = \mathcal{R}^{-1}(s)X, s], \tag{38}$$

and from Eq. (A6), we have

$$\frac{\partial f(X,s)}{\partial s} = \frac{\partial \rho(X_0,s)}{\partial s} - \Big[\mathcal{R}^{-1}(s)\, M(s)\, X\Big]_i \, \frac{\partial \rho(X_0,s)}{\partial (X_0)_i} \tag{39}$$

and

$$\frac{\partial f(X,s)}{\partial X_i} = [\mathcal{R}^{-1}(s)]_{ji} \, \frac{\partial \rho(X_0,s)}{\partial (X_0)_j}. \tag{40}$$

Therefore the Vlasov equation for $\rho(X_0,s)$ can be obtained from Eq. (37):

$$\frac{\partial \rho(X_0,s)}{\partial s} + [\, \mathcal{R}^{-1}(s)\, F^{[f]}(\mathcal{R}(s)X_0,s)\,]_i \, \frac{\partial \rho(X_0,s)}{\partial (X_0)_i} = 0, \tag{41}$$

which is equivalent to the integral equation

$$\rho(X_0,s) = \rho(X_0,0) - \int_0^s ds_\tau \, [\mathcal{R}^{-1}(s_\tau) F^{[f]}(\mathcal{R}(s_\tau)X_0,s_\tau)]_i \, \frac{\partial \rho(X_0,s_\tau)}{\partial (X_0)_i}. \tag{42}$$

Denoting $X_\tau = \mathcal{R}(s_\tau)X_0$, and using

$$\frac{\partial \rho(X_0,s_\tau)}{\partial (X_0)_i} = [\mathcal{R}(s_\tau)]_{ji} \frac{\partial f(X_\tau,s_\tau)}{\partial (X_\tau)_j}, \tag{43}$$

we can convert Eq. (42) into an integral equation for $f(X,s)$

$$f(X,s) = f^{(0)}(X,s) - \int_0^s ds_\tau [F^{[f]}(X_\tau,s_\tau)]_i \, \frac{\partial f(X_\tau,s_\tau)}{\partial (X_\tau)_i}, \tag{44}$$

for $X_\tau = [\mathcal{R}(s_\tau \to s)]^{-1} X$ with $\mathcal{R}(s_\tau \to s) = \mathcal{R}(s)\mathcal{R}^{-1}(s_\tau)$. Here

$$f^{(0)}(X,s) = \rho(\mathcal{R}^{-1}(s)X,0) \tag{45}$$

gives the phase space distribution under nominal optical transport, and the integral in Eq. (44) gives the impact of the effective forces during $0 \le s_\tau \le s$ on $f(X,s)$. For small effective CSR forces, the perturbation on the nominal phase space distribution is yielded from the first iteration of Eq. (44):

$$f(X,s) \simeq f^{(0)}(X,s) - \int_0^s ds_\tau \, [F^{[f^{(0)}]}(X_\tau,s_\tau)]_i \, \frac{\partial f^{(0)}(X_\tau,s_\tau)}{\partial (X_\tau)_i}. \tag{46}$$

The evolution of a small longitudinal density perturbation on a stable distribution, and the relations of our phase space distribution $f(X,s)$ with measurable quantities, are given in Appendix B.

4. RETARDED POTENTIALS FOR A 2D ENERGY-CHIRPED BUNCH ON A CIRCULAR ORBIT

In the previous sections, the equations of motion and the phase space evolution equation are laid out for a charge distribution being transported along a curved orbit by an external EM field, and the perturbation of the charge phase space distribution is expressed in terms of the effective CSR forces, or the curvature-induced bunch collective interaction forces. Here we are interested in studying the effect of bunch tilting (x-z correlation) on the effective CSR forces when an energy-chirped bunch (δ-z correlation) goes through a dispersive region (x-δ correlation) for processes such as bunch compression in a magnetic chicane.

Our first step is to analyze the retarded potentials for an energy-chirped bunch under zeroth order approximation. This result will then be used in the following section for the analysis of $F^{[f^{(0)}]}$ in Eq. (46), which arises when one takes the first order iteration in solving Eq. (44). Here the zeroth order approximation implies that an initial phase space distribution is transported through the chicane under nominal optical transport

$$f(X_r, s_r) = f^{(0)}(X_r, s_r) = \rho[\mathcal{R}^{-1}(s_r)X_r, 0] = \rho(X_{r0}, 0), \tag{47}$$

where X_r and s_r are respectively the phase space variables and pathlength parameter for the retarded distribution $f(X_r, s_r)$ in Eq. (28), and $X_{r0} = \mathcal{R}^{-1}(s_r)X_r$. The effect of energy chirping is included in the parameter u in $\mathcal{R}(s)$ of Eq. (A5). For simplicity, here we consider only a two-dimensional charge distribution in the $y = 0$ plane, with $X = (x, x', 0, 0, z, \delta_H)^T$ and $X_r = (x_r, x_r', 0, 0, z_r, \delta_{Hr})^T$. With the change of variables from X_r to X_{r0},

$$X_r = [x_r, x_r', 0, 0, z_r, (\delta_H)_r]^T = \mathcal{R}(s_r)X_{r0}, \tag{48}$$

and using Eq. (47), the collective potentials in Eq. (28) become

$$\left\{ \begin{array}{rcl} \tilde{\Phi}(x, z, s) &=& \dfrac{2\beta_0 r_e}{\gamma_0} \displaystyle\int ds_r \int dX_{r0}\, \rho(X_{r0}, 0) w_0(X_{r0}, s_r)\, \delta(P)\, \theta(Q) \\ \tilde{\mathbf{A}}(x, z, s) &=& \dfrac{2\beta_0^2 r_e}{\gamma_0} \displaystyle\int ds_r \int dX_{r0}\, \rho(X_{r0}, 0) \mathbf{w}(X_{r0}, s_r)\, \delta(P)\, \theta(Q) \end{array} \right. , \tag{49}$$

where

$$\left\{ \begin{array}{rcl} w_0(X_{r0}, s_r) &=& 1 + \kappa_0(s_r) x_r(X_{r0}, s_r) \\ \mathbf{w}(X_{r0}, s_r) &=& [1 + \kappa_0(s_r) x_r(X_{r0}, s_r)]\mathbf{e}_s(s_r) + x_r'(X_{r0}, s_r)\, \mathbf{e}_x(s_r) \end{array} \right. , \tag{50}$$

and P, Q in Eq. (49) are

$$\begin{cases} P(x,z,s,X_{r0},s_r) &= \{(s-s_r)-[z-z_r(X_{r0},s_r)]\}^2 - \beta_0^2\{\mathbf{r}(x,s)-\mathbf{r}[x_r(X_{r0},s_r),s_r]\}^2 \\ Q(x,z,s,X_{r0},s_r) &= (s-s_r)-[z-z_r(X_{r0},s_r)] \end{cases}. \tag{51}$$

In Eq. (49), $\delta(P)$ may contain both retarded and advanced solutions, and $\theta(Q)$ ensures the selection of only the retarded solution. For interactions within a single magnetic bend when both s and s_r are on the same circular orbit with radius $R_0 = 1/\kappa_0$, one has $[\mathbf{r}(x,s) - \mathbf{r}(x_r,s_r)]^2$ in Eq. (51) as

$$[\mathbf{r}(x,s)-\mathbf{r}(x_r,s_r)]^2 = \left(1+\frac{x}{R_0}\right)\left(1+\frac{x_r}{R_0}\right)\left(2R_0\sin\frac{s-s_r}{2R_0}\right)^2 + (x-x_r)^2. \tag{52}$$

As a result of the initial linear energy chirping along the bunch length, in dispersive regions, the bunch has a horizontal-longitudinal correlation in configuration space. Using Eq. (48) for the source particle, with $\mathcal{R}_{ij}(s_r)$ given in Eq. (A5), we write

$$z_r(X_{r0},s_r) = z_{r1} + \mathcal{R}_{55}(s_r)z_{r0}, \quad x_r(X_{r0},s_r) = x_{r1} + \mathcal{R}_{15}(s_r)z_{r0} \tag{53}$$

with

$$z_{r1} = \mathcal{R}_{5j}(s_r)(Z_{r0})_j, \quad x_{r1} = \mathcal{R}_{1j}(s_r)(Z_{r0})_j \tag{54}$$

for Z_{r0} representing the intrinsic transverse and energy offsets:

$$Z_{r0} = [x_{r0}, x'_{r0}, 0, 0, 0, (\delta_H)_{r0}]^T. \tag{55}$$

Similarly, we write

$$z(X_0,s) = z_1 + \mathcal{R}_{55}(s)z_0, \quad x(X_0,s) = x_1 + \mathcal{R}_{15}(s)z_0 \tag{56}$$

with

$$z_1 = \mathcal{R}_{5j}(s)(Z_0)_j, \quad x_1 = \mathcal{R}_{1j}(s)(Z_0)_j \tag{57}$$

for

$$Z_0 = [x_0, x'_0, 0, 0, 0, (\delta_H)_0]^T. \tag{58}$$

We then denote the bunch tilting factor as

$$\xi(s) = \frac{\mathcal{R}_{15}(s)}{\mathcal{R}_{55}(s)} = \frac{uR_{16}(s)}{1+uR_{56}(s)}, \qquad \xi_r = \xi(s_r). \tag{59}$$

For a linear bunch with zero emittance and uncorrelated energy spread, Eq. (53) yields the slope of the bunch at s_r: $x_r/z_r = \xi_r$. The "on-orbit" case for a projected bunch studied earlier [5–7] corresponds to zero energy chirping $u = 0$, when the tilting factor $\xi(s) = 0$ for all s.

For a test particle at (x, z, s) to receive fields generated by source particles at s_r, knowing the source particles' intrinsic transverse and energy offset Z_{r0}, we need to solve the initial longitudinal position z_{r0} of the source particles by evaluating $\delta(P)$ in Eq. (49). Let us define

$$\Delta s = s - s_r, \quad \Delta z = z - \mathcal{R}_{55}(s_r)z_{r0}, \quad \text{and} \quad (\bar{z}, \Delta\bar{z}, \Delta\bar{s}) = \frac{1}{|R_0|}(z, \Delta z, \Delta s) \tag{60}$$

for a constant curvature $\kappa_0 = 1/R_0$. We also define $\hat{x}$ and $\hat{\xi}$ as

$$\hat{x}_r = \frac{x_r}{R_0}, \quad \hat{x} = \frac{x}{R_0}, \quad \hat{\xi}_r(s) = \xi(s_r)\frac{R_0}{|R_0|}, \quad \hat{\xi}(s) = \xi(s)\frac{R_0}{|R_0|}, \tag{61}$$

which are invariant when the beamline is under mirror reflection $R_0 \to -R_0$ and $x \to -x$. Thus when both s and s_r are inside the same magnetic bend with radius R_0, using Eq. (52), P in Eq. (51) becomes a quadratic function of $\Delta\bar{z}$

$$P = R_0^2\left(a\Delta\bar{z}^2 - 2b\Delta\bar{z} + c\right) \tag{62}$$

with a, b, c given in Appendix C in terms of (x, z, s) and (Z_{r0}, s_r). Thus we have for $b^2 - ac \geq 0$

$$\delta(P) = \frac{\delta(\Delta\bar{z} - \Delta\bar{z}^{(+)}) + \delta(\Delta\bar{z} - \Delta\bar{z}^{(-)})}{2R_0^2\sqrt{b^2 - ac}} \tag{63}$$

with $\Delta\bar{z}^{(\pm)}$ the roots for $P = 0$:

$$\Delta\bar{z}^{(\pm)}(x, z, s; Z_{r0}, s_r) = \frac{b \pm \sqrt{b^2 - ac}}{a}. \tag{64}$$

This leads us to the solution of $\bar{z}_{r0}$

$$\delta(P) = \frac{\delta\big(\bar{z}_{r0} - \bar{z}_{r0}^{(+)}\big) + \delta\big(\bar{z}_{r0} - \bar{z}_{r0}^{(-)}\big)}{2R_0^2\left|\mathcal{R}_{55}(s_r)\right|\sqrt{b^2 - ac}} \qquad (b^2 - ac > 0), \tag{65}$$

in which

$$\bar{z}_{r0}^{(\pm)}(x, z, s; Z_{r0}, s_r) = \frac{\bar{z} - \Delta\bar{z}^{(\pm)}}{\mathcal{R}_{55}(s_r)} = \frac{a\bar{z} - b \mp \sqrt{b^2 - ac}}{a\mathcal{R}_{55}(s_r)}. \tag{66}$$

The expression for $\bar{z}_{r0}^{(\pm)}$ can be obtained by substituting Eqs. (C1) and (C8) into Eq. (66):

$$\bar{z}_{r0}^{(\pm)} = \frac{-\Delta\bar{s}(1 - \hat{\xi}_r\Delta\bar{s}/2) - (\hat{\xi}_r\hat{x} - \bar{z}) - (\bar{z}_{r1} - \hat{\xi}_r\hat{x}_{r1}) \mp \sqrt{\omega_0 + \omega_1}}{(1 - \hat{\xi}_r^2)\mathcal{R}_{55}(s_r)}. \tag{67}$$

In cases of moderate or strong tilting, $\hat{\xi}_r \gg 1$, it is convenient to use

$$\hat{\chi}(s) = \frac{1}{\hat{\xi}(s)}, \quad \text{and} \quad \hat{\chi}_r = \frac{1}{\hat{\xi}_r}. \tag{68}$$

With both the nominator and denominator multiplied by $\hat{\chi}_r$, Eq. (67) becomes

$$\bar{z}_{r0}^{(\pm)} = \frac{\Delta\bar{s}(\hat{\chi}_r - \Delta\bar{s}/2) + (\hat{x} - \hat{\chi}_r\bar{z}) - (\hat{x}_{r1} - \hat{\chi}_r\bar{z}_{r1}) \pm \mathrm{sign}(\hat{\chi}_r)\sqrt{(\omega_0+\omega_1)\hat{\chi}_r^2}}{(1-\hat{\chi}_r^2)u\hat{R}_{16}(s_r)}, \tag{69}$$

with $(\omega_0+\omega_1)\hat{\chi}_r^2$ given by Eq. (C9), and $\hat{R}_{16}(s_r) = R_{16}(s_r)\cdot\mathrm{sign}(\kappa_0)$.

We now assume the initial normalized phase space distribution $\rho(X_{r0},0)$ takes the form

$$\rho(X_{r0},0) = N\rho_{\mathrm{in}}(Z_{r0})\,\lambda_0(z_{r0}), \tag{70}$$

with N the total number of electrons, $\lambda_0(z_{r0})$ the initial longitudinal charge density distribution function and $\rho_{\mathrm{in}}(Z_{r0})$ the initial *intrinsic* transverse and energy distribution function,

$$\int \rho_{in}(Z_{r0})dZ_{r0} = 1, \quad \text{and} \quad \int \lambda_0(z_{r0})dz_{r0} = 1. \tag{71}$$

For simplicity, let us consider only the "steady-state" situation [17] when the potentials in Eq. (49) on a test particle at s in a bending magnet are contributed mainly from $s_r \in [s_1, s_2]$, or $\Delta s \in [s-s_2, s-s_1]$, with s_1 and s_2 the pathlengths at the entrance and exit of the same bending magnet. This geometry implies

$$\begin{aligned}
\mathbf{e}_x(s_r)\cdot\mathbf{e}_s(s) &= -\mathbf{e}_s(s_r)\cdot\mathbf{e}_x(s) = \sin\frac{s-s_r}{R_0},\\
\mathbf{e}_s(s_r)\cdot\mathbf{e}_s(s) &= \mathbf{e}_x(s_r)\cdot\mathbf{e}_x(s) = \cos\frac{s-s_r}{R_0}.
\end{aligned}$$

We now apply Eqs. (65) and (70) to Eq. (49), and let $dZ_{r0} = dx_{r0}dx'_{r0}d(\delta_H)_{r0}$. Defining

$$W(x,z,s;Z_{r0},s_r) = \frac{\rho_{\mathrm{in}}(Z_{r0})}{|\mathcal{R}_{55}(s_r)|\sqrt{b^2-ac}}, \tag{72}$$

one obtains from Eq. (49) the potential terms shown in Eq. (24)

$$\left\{\begin{aligned}
\tilde{\Phi}(x,z,s) &= \frac{\beta_0 N r_e}{\gamma_0|R_0|}\Big[\int_{\Omega^+} ds_r dZ_{r0}\, W(x,z,s;Z_{r0},s_r)\,\lambda(z_{r0}^{(+)})\,H_0(z_{r0}^{(+)},Z_{r0},s_r)\\
&\qquad + \int_{\Omega^-} ds_r dZ_{r0}\, W(x,z,s;Z_{r0},s_r)\,\lambda(z_{r0}^{(-)})\,H_0(z_{r0}^{(-)},Z_{r0},s_r)\Big]\\
[\tilde{A}_s - \beta_0\tilde{\Phi}] &= \frac{\beta_0^2 N r_e}{\gamma_0|R_0|}\Big[\int_{\Omega^+} ds_r dZ_{r0}\, W(x,z,s;Z_{r0},s_r)\,\lambda(z_{r0}^{(+)})\,H_s(z_{r0}^{(+)},Z_{r0},s_r)\\
&\qquad + \int_{\Omega^-} ds_r dZ_{r0}\, W(x,z,s;Z_{r0},s_r)\,\lambda(z_{r0}^{(-)})\,H_s(z_{r0}^{(-)},Z_{r0},s_r)\Big]\\
\tilde{A}_x(x,z,s) &= \frac{\beta_0^2 N r_e}{\gamma_0|R_0|}\Big[\int_{\Omega^+} ds_r dZ_{r0}\, W(x,z,s;Z_{r0},s_r)\,\lambda(z_{r0}^{(+)})\,H_x(z_{r0}^{(+)},Z_{r0},s_r)\\
&\qquad + \int_{\Omega^-} ds_r dZ_{r0}\, W(x,z,s;Z_{r0},s_r)\,\lambda(z_{r0}^{(-)})\,H_x(z_{r0}^{(-)},Z_{r0},s_r)\Big]
\end{aligned}\right. \tag{73}$$

with $z_{r0}^{(\pm)} = |R_0|\bar{z}_{r0}^{(\pm)}$ given in Eq. (66), and

$$\begin{cases} H_0(z_{r0}, Z_{r0}, s_r) = 1 + \dfrac{\mathcal{R}_{15}(s_r)z_{r0} + \mathcal{R}_{1j}(s_r)(Z_{r0})_j}{R_0} \\ H_s(z_{r0}, Z_{r0}, s_r) = [\mathcal{R}_{25}(s_r)z_{r0} + \mathcal{R}_{2j}(s_r)(Z_{r0})_j]\sin\dfrac{\Delta s}{R_0} - H_0(z_{r0}, Z_{r0}, s_r)(1-\cos\Delta\bar{s}) \\ H_x(z_{r0}, Z_{r0}, s_r) = -H_0(z_{r0}, Z_{r0}, s_r)\sin\dfrac{\Delta s}{R_0} + [\mathcal{R}_{25}(s_r)z_{r0} + \mathcal{R}_{2j}(s_r)(Z_{r0})_j]\cos\Delta\bar{s} \end{cases} . \tag{74}$$

In Eq. (73), the range of phase space integration, $\Omega^{\pm}$, are set to ensure the existence of solutions for Eq. (63) and to exclude the advanced solution:

$$\Omega^{\pm} : \{b^2 - ac \geq 0 \quad \text{and} \quad Q^{(\pm)} = \Delta s + z_{r1} - \Delta z^{(\pm)} \geq 0\}, \tag{75}$$

with z_{r1} and $\Delta z^{(\pm)}$ given by Eqs. (54) and (64) respectively. Discussions of conditions in Eq. (75) can be found in Appendices C2 and C3.

5. RETARDATION SOLUTIONS FOR VARIOUS LEVELS OF BUNCH TILTING

In Eq. (65) of Sec. 4, the retardation relation is solved for the CSR interaction on a circular orbit with curvature κ_0 and radius $R_0 = 1/\kappa_0$, where the initial longitudinal position of a source particle z_{r0} is solved in terms of the bunch internal coordinates (x, z, s) of the test particle, the initial intrinsic offset Z_{r0}, and the retarded pathlength s_r of the source particle. With $\bar{z}_0$ in Eq. (56) denoting the initial internal longitudinal position of the test particle, and $\sigma_z(s)$ and $\sigma_x(s)$ the longitudinal and transverse rms bunch size at pathlength s respectively, we can now list the retardation solutions for various levels of bunch tilt under the assumption of Eq. (C3).

5.1. Nontilted bunch: $u = 0$, $\xi_r = 0$

Setting $\xi_r = 0$ and $\mathcal{R}_{55}(s_r) = 1$ in Eq. (67), and assuming

$$(\hat{x} - \hat{x}_{r1})^2/(\Delta\bar{s})^2 \ll \hat{x} + \hat{x}_{r1}, \quad \text{or} \quad \frac{\sigma_x(s)}{|R_0|} \ll (\Delta\bar{s})^2, \tag{76}$$

we get the retardation solutions $(\bar{z}_{r0})_1$ and $(\bar{z}_{r0})_2$ from Eq. (67) for $\Delta\bar{s} > 0$ (back-front interaction),

$$(\bar{z}_{r0})_1 = \bar{z}_{r0}^{(-)} \simeq (\bar{z} + \hat{x}\Delta\bar{s}/2) - \Delta\bar{s}^3/24 - (\bar{z}_{r1} - \hat{x}_{r1}\Delta\bar{s}/2), \tag{77}$$

and for $\Delta\bar{s} < 0$ (front-back interaction)

$$(\bar{z}_{r0})_2 = \bar{z}_{r0}^{(-)} \simeq (\bar{z} + \hat{x}|\Delta\bar{s}|/2) + 2|\Delta\bar{s}| - |\Delta\bar{s}|^3/24 - (\bar{z}_{r1} - \hat{x}_{r1}|\Delta\bar{s}|/2). \tag{78}$$

For a bunch with the intrinsic horizontal emittance and uncorrelated (canonical) energy spread small enough to satisfy

$$O(\hat{x}_1, \bar{z}_1, \hat{x}_{r1}, \bar{z}_{r1}) \ll |\Delta\bar{s}^3|, \tag{79}$$

the solutions in Eqs. (77) and (78) reduce to

$$\bar{z}_0 - (\bar{z}_{r0})_1 \simeq \Delta\bar{s}^3/24 \qquad (\Delta\bar{s} > 0) \tag{80}$$

and

$$(\bar{z}_{r0})_2 - \bar{z}_0 \simeq 2|\Delta\bar{s}| - |\Delta\bar{s}|^3/24 \qquad (\Delta\bar{s} < 0). \tag{81}$$

Here Eq. (80) indicates that for a nontilted bunch, the back-front ($s_r < s$) interaction is for a test particle at bunch head to receive interaction generated by a source particle at bunch tail $(\bar{z}_{r0})_1 < \bar{z}_0$. On the contrary, Eq. (81) indicates that the front-back ($s_r > s$) interaction is for a test particle at bunch tail to receive interaction from a source particle at bunch head $(\bar{z}_{r0})_2 > \bar{z}_0$. Note $(\bar{z}_{r0})^{(+)}$ is excluded because in Eq. (75) $Q^{(+)} < 0$ (see Appendix C.3).

5.2. Small Tilt

Let us assume $O(\hat{x}) \ll (\Delta\bar{s})^2 < 1$ and consider a small tilt when $|ac| \ll b^2$. The retardation solutions can be obtained by applying this small tilt condition to Eq. (67). For $\Delta\bar{s} > 0$, the first solution yields

$$(\bar{z}_{r0})_1 = \bar{z}_{r0}^{(-)} \simeq \frac{(\bar{z} + \hat{x}\Delta\bar{s}/2) - \Delta\bar{s}^3/24 - (\bar{z}_{r1} - \hat{x}_{r1}\Delta\bar{s}/2)}{\mathcal{R}_{55}(s_r)(1 - \xi_r\Delta\bar{s}/2)}, \tag{82}$$

which reduces to Eq. (77) when $u = 0$. For $\xi_r^2 < 1$, the second solution exists for $\Delta\bar{s} < 0$,

$$(\bar{z}_{r0})_2 = (\bar{z}_{r0}^{(-)}) = \frac{1}{\mathcal{R}_{55}(s_r)} \left[\bar{z} + \frac{2|\Delta\bar{s}|}{1 - \xi_r^2}\left(1 - \hat{\xi}_r\frac{\Delta\bar{s}}{2}\right)\right] + \cdots \quad . \tag{83}$$

According to Appendix C.3, as ξ_r^2 increases to a value bigger than 1, this second solution turns to be back-front interaction. Namely, for $\xi_r^2 > 1$, the second solution exists for $\Delta\bar{s} > 0$,

$$(\bar{z}_{r0})_2 = (\bar{z}_{r0}^{(+)}) = \frac{1}{\mathcal{R}_{55}(s_r)} \left[\bar{z} + \frac{2\Delta\bar{s}}{\xi_r^2 - 1}\left(1 - \hat{\xi}_r\frac{\Delta\bar{s}}{2}\right)\right] + \cdots \quad . \tag{84}$$

Remark that when $\xi_r^2 \to 1$, only $(\bar{z}_{r0})_1$ exists while $(\bar{z}_{r0})_2$ runs to infinity.

5.3. Moderate Tilt

It is now more convenient to use $\chi_r = 1/\xi_r$ instead of ξ_r, where $\chi_r^2 \ll 1$. Due to $Q^{(\pm)}$ in Appendix C.3, the retardation solutions in Eq. (69) exist only for $\Delta\bar{s} > 0$ when $\xi_r^2 > 1$. Under the approximation of Eq. (C10) with $\Delta\bar{s}$ given by the "steady-state" condition in Eq. (125), we have the criterion for a moderate tilt

$$O\left(\frac{[\hat{\chi}\mathcal{R}_{1j}(s) - \mathcal{R}_{5j}(s)]\sigma_{0j}}{|R_0|\hat{\chi}^2}\right) \ll O([\bar{\sigma}_z(s)]^{1/3}) \tag{85}$$

with $\bar{\sigma}_z(s)$ given in Eq. (115). Thus Eq. (69) reduces to

$$(\bar{z}_{r0})_1 \simeq \frac{\left(\bar{z} + \hat{x}\frac{\Delta\bar{s}}{2}\right) - \left(\bar{z}_{r1} - \hat{x}_{r1}\frac{\Delta\bar{s}}{2}\right) - \frac{\Delta\bar{s}^3}{24}(1-\hat{\chi}_r^2)}{u\hat{R}_{16}(s_r)(1-\hat{\chi}_r^2)\left(\hat{\chi}_r - \frac{\Delta\bar{s}}{2}\right)}, \tag{86}$$

$$(\bar{z}_{r0})_2 \simeq \frac{1}{u\hat{R}_{16}(s_r)(1-\hat{\chi}_r^2)\left(\hat{\chi}_r - \frac{\Delta\bar{s}}{2}\right)}\left\{2\Delta\bar{s}\left(\hat{\chi}_r - \frac{\Delta\bar{s}}{2}\right)^2 + \frac{\Delta\bar{s}^3}{24}(1-\hat{\chi}_r^2)\right. \tag{87}$$

$$\left. + \left[\hat{x}\left(2\hat{\chi}_r - \frac{3\Delta\bar{s}}{2}\right) - \bar{z}\right] + \left[\bar{z}_{r1} - \hat{x}_{r1}\left(2\hat{\chi}_r - \frac{\Delta\bar{s}}{2}\right)\right]\right\}. \tag{88}$$

Furthermore, with Eq. (56) for $\bar{z}$ and x, and results in Appendix D for $O[R_0\Delta\bar{s}^2/\hat{R}_{16}(s)] \ll 1$, the above retarded solutions become

$$(\bar{z}_{r0})_1 \simeq \bar{z}_0 - \frac{1}{\mathcal{R}_{55}(s)[1+\Delta\bar{s}/2\hat{\chi}(s)]}\left[\frac{\Delta\bar{s}^3}{24} - \left(\bar{z}_1 + \hat{x}_1\frac{\Delta\bar{s}}{2}\right) + \left(\bar{z}_{r1} - \hat{x}_{r1}\frac{\Delta\bar{s}}{2}\right)\right], \tag{89}$$

$$(\bar{z}_{r0})_2 \simeq \bar{z}_0 + \frac{1}{\mathcal{R}_{55}(s)[1+\Delta\bar{s}/2\hat{\chi}(s)]}\left\{\frac{\Delta\bar{s}^3}{24} + 2\Delta\bar{s}\left(\hat{\chi}_r - \frac{\Delta\bar{s}}{2}\right)^2\right.$$

$$\left. - \left[\bar{z}_1 - \hat{x}_1\left(2\hat{\chi} + \frac{\Delta\bar{s}}{2}\right)\right] + \left[\bar{z}_{r1} - \hat{x}_{r1}\left(2\hat{\chi} + \frac{3\Delta\bar{s}}{2}\right)\right]\right\}. \tag{90}$$

For small initial intrinsic spread satisfying Eq. (79), we have from Eqs. (89) and (90)

$$\bar{z}_0 - (\bar{z}_{r0})_1 \simeq \frac{\Delta\bar{s}^3/24}{\mathcal{R}_{55}(s)(1+\Delta\bar{s}/2\hat{\chi})}, \tag{91}$$

$$(\bar{z}_{r0})_2 - \bar{z}_0 \simeq \frac{1}{\mathcal{R}_{55}(s)(1+\Delta\bar{s}/2\hat{\chi})}\left[2\Delta\bar{s}\,\hat{\chi}^2\left(1+\frac{\Delta\bar{s}}{2\hat{\chi}}\right)^2 + \frac{\Delta\bar{s}^3}{24}\right]. \tag{92}$$

Note the tail-head interaction now has a factor $(1+\Delta\bar{s}/2\hat{\chi})$ in Eq. (91) compared to the usual nontilted bunch case in Eq. (80). Unlike in Eq. (81), where the head-tail interaction is generated at $s_r > s$, for a tilted line bunch with $\xi^2 > 1$, the head-tail interaction now is generated at $s_r < s$, with $(z_{r0})_2 - z_0$ smaller for larger tilt $\xi \gg 1$, in comparison with $(z_{r0})_2 - z_0$ in Eq. (81). An illustration of the interaction of two source particles with a test particle on a circular orbit for a tilted bunch is shown in Fig. 3.

6. EFFECTIVE LONGITUDINAL CSR FORCE FOR A LINEAR ENERGY-CHIRPED BUNCH

We now study the "steady-state" effective longitudinal CSR force for a moderately tilted bunch. For a 2D energy-chirped bunch on a circular orbit, the effective longitudinal CSR force can be obtained by applying $\tilde{A}_s - \beta_0\tilde{\Phi}$ and $\tilde{A}_x$ in Eq. (31) to $\tilde{F}_H$ in Eq. (24):

$$\begin{aligned}\tilde{F}_H^{[f^{(0)}]}(X,s) &= \frac{\partial}{\partial z}\left[(1+\kappa_0 x)(\tilde{A}_s - \beta_0\tilde{\Phi}) + x'\tilde{A}_x\right] \\ &= \frac{\beta_0 N r_e}{\gamma_0}\int ds_r dX_r\, W_H(X,s,X_r,s_r)\,\frac{\partial f(X_r,s_r)}{\partial z_r} \\ &\quad\times\frac{\delta\left[\,(s-s_r)-(z-z_r)-\beta_0|\mathbf{r}(x,s)-\mathbf{r}(x_r,s_r)|\,\right]}{|\mathbf{r}(x,s)-\mathbf{r}(x_r,s_r)|},\end{aligned} \tag{93}$$

with

$$W_H[X,s,X_r,s_r] = -(1+\hat{x})(1+\hat{x}_r)2\sin^2\frac{\Delta\bar{s}}{2}+(\hat{x}'_r-\hat{x}'+\hat{x}\hat{x}'_r-\hat{x}_r\hat{x}')\sin\Delta\bar{s}+\hat{x}'\hat{x}'_r\cos\Delta\bar{s}. \tag{94}$$

Next, with the zeroth order approximation $f(X_r,s_r) = \rho[\mathcal{R}^{-1}(s_r)X_r,0]$ and the change of variables $X_{r0} = \mathcal{R}^{-1}(s_r)X_r$, and $[\mathcal{R}^{-1}(s_r)]_{i5} = (0,0,1,-u)^T$, we have in Eq. (93)

$$\frac{\partial f(X_r,s_r)}{\partial z_r} = \frac{\partial\rho(X_{r0},0)}{\partial z_{r0}} - u\frac{\partial\rho_0(X_{r0})}{\partial(\delta_H)_{r0}}. \tag{95}$$

Moreover, with

$$\frac{\delta\left[(s-s_r)-(z-z_r)-\beta_0|\mathbf{r}(x,s)-\mathbf{r}(x_r,s_r)|\right]}{\beta_0|\mathbf{r}(x,s)-\mathbf{r}(x_r,s_r)|} = 2\delta(P)\;\theta[(s-s_r)-(z-z_r)] \tag{96}$$

for $\delta(P)$ given in Eq. (65), Z_{r0} in Eq. (55) and $\rho(X_{r0},0)$ given in Eq. (70), Eq. (93) becomes

$$\tilde{F}_H^{[f^{(0)}]}(X,s) = \frac{\beta_0^2 N r_e}{\gamma_0|R_0|}\left(\int_{\Omega^-} ds_r dZ_{r0}\; g[z_{r0}=(z_{r0})_1] + \int_{\Omega^+} ds_r dZ_{r0}\; g[z_{r0}=(z_{r0})_2]\right), \tag{97}$$

where $(z_{r0})_{1,2} = |R_0|(\bar{z}_{r0})_{1,2}$ are given by Eqs. (89) and (90), and the integrand g in Eq. (97) is

$$g(X,s,Z_{r0},z_{r0},s) = \frac{W_H[X,s,X_r(Z_{r0},z_{r0}),s_r]}{|\mathcal{R}_{55}(s_r)|\sqrt{b^2-ac}}\left[\rho_{in}(Z_{r0})\frac{\partial\lambda_0(z_{r0})}{\partial z_{r0}} - u\lambda_0(z_{r0})\frac{\partial\rho_{in}(Z_{r0})}{\partial(\delta_H)_{r0}}\right]. \tag{98}$$

Here W_H in Eq. (98) is given by Eq. (94), with

$$\begin{aligned}\hat{x}'_r &= x'_r\,\mathrm{sign}(\kappa_0) = [\mathcal{R}_{25}(s_r)z_{r0} + \mathcal{R}_{2j}(s_r)(Z_{r0})_j]\;\mathrm{sign}(\kappa_0), \\ \hat{x}' &= x'\,\mathrm{sign}(\kappa_0) = [\mathcal{R}_{25}(s)z_0 + \mathcal{R}_{2j}(s)(Z_0)_j]\;\mathrm{sign}(\kappa_0).\end{aligned}$$

In the following we use Eq. (97) to study the effective longitudinal force for a moderately tilted bunch with a simple initial phase space distribution.

6.1. Modulated Initial Longitudinal Distribution

Assume the longitudinal density distribution in Eq. (70) is the real part of

$$\frac{1}{\sqrt{2\pi}\sigma_{z0}}\exp\left(-\frac{z_0^2}{2\sigma_{z0}^2}\right)\mathrm{Re}\left[e^{i(k_0z_0+\phi_0)}\right] \tag{99}$$

for $k_0 \gg \sigma_{z0}^{-1}$, so let us consider

$$\lambda_0(z_0) = \frac{1}{\sqrt{2\pi}\sigma_{z0}}\exp\left(-\frac{z_0^2}{2\sigma_{z0}^2}+ik_0z_0\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,\exp\left[-\frac{(k-k_0)^2\sigma_{z0}^2}{2}+ikz_0\right], \tag{100}$$

and an intrinsic distribution with transverse emittance ϵ_{x0} and (canonical) energy spread σ_{H0}:

$$\rho_{in}(Z_0) = \frac{1}{2\pi\epsilon_{x0}}\exp\left(-\frac{\tilde{x}_0^2}{2}-\frac{\tilde{x}_0'^2}{2}\right)\frac{1}{\sqrt{2\pi}\sigma_{H0}}\exp\left(-\frac{(\tilde{\delta}_{H0})^2}{2}\right), \tag{101}$$

where

$$\tilde{x}_0 = \frac{x_0}{\sigma_{x0}},\qquad \tilde{x}_0' = \frac{x_0'}{\sigma_{x0'}},\qquad \tilde{\delta}_{H0} = \frac{\delta_{H0}}{\sigma_{H0}} \tag{102}$$

for $\sigma_{x0} = \sqrt{\beta_{x0}\epsilon_{x0}}$ and $\sigma_{x0'} = \sqrt{\epsilon_{x0}/\beta_{x0}}$, with β_{x0} the initial transverse beta function at $s = 0$. We now calculate the first integral in Eq. (97), which is mainly contributed from the $-2\sin^2(\Delta\bar{s}/2)$ term in W_H of Eq. (94). Under the approximation in Eq. (126), we have

$$\begin{aligned}(\tilde{F}_H)_1 &= \frac{\beta_0^2Nr_e}{\gamma_0|R_0|}\int_{\Omega^-}ds_rdZ_{r0}\;g[z_{r0}=(z_{r0})_1]\\ &\simeq \frac{\beta_0^2Nr_e}{2\pi\gamma_0}\int_0^{\infty}\frac{d\Delta\bar{s}}{|\mathcal{R}_{55}(s)|}\frac{-\Delta\bar{s}/2}{\sqrt{\Lambda(\Delta\bar{s},\hat{\chi})}}\int_{-\infty}^{\infty}dk\,\exp\left(-\frac{(k-k_0)^2\sigma_{z0}^2}{2}\right)I_A(s,\Delta\bar{s},k)\end{aligned} \tag{103}$$

for

$$I_A(s,\Delta\bar{s},k) = \int_{-\infty}^{\infty}dZ_{r0}\left(ik+u\frac{(\tilde{\delta}_H)_{r0}}{\sigma_{H0}}\right)\rho_{in}(Z_{r0})\,\exp[ik\,(z_{r0})_1], \tag{104}$$

where

$$|\mathcal{R}_{55}(s_r)||\sqrt{b^2-ac}| \simeq |\mathcal{R}_{55}(s)|\Delta\bar{s}\sqrt{\Lambda(\Delta\bar{s},\hat{\chi})} \tag{105}$$

is used based on Eqs. (C12) and (D5), with

$$\Lambda(\Delta\bar{s},\hat{\chi}) = \left(1+\frac{\Delta\bar{s}}{2\hat{\chi}}\right)^2+\frac{(\Delta\bar{s})^2}{12\hat{\chi}^2}, \tag{106}$$

and $(\tilde{z}_{r0})_1$ is given in Eq. (89). Here in Eq. (103), the lower integral limit for $\Delta\bar{s}$ is extended to $\Delta\bar{s} = 0$, assuming that $\sigma_{0j}(j = 1, 2, 6)$ are small enough that small $\Delta\bar{s}$ beyond the range specified in Eq. (C10) has negligible contribution to Eq. (103). Meanwhile, the upper integral limit for $\Delta\bar{s}$ is extended to infinity, since for the "steady-state" interaction discussed here, there is a negligible contribution from $\Delta\bar{s} \gg [\bar{\sigma}_z(s)]^{1/3}$ (see Eq. (125)).

Next, let us define

$$\Pi(s, \Delta\bar{s}, k) = \exp\left[-\frac{k^2}{2}\Sigma^2(s, \Delta\bar{s})\right] \tag{107}$$

for

$$\Sigma^2(s, \Delta\bar{s}) = \sum_{j=1,2,6} \left(\frac{\mathcal{R}_{5j}(s_r) - \hat{\mathcal{R}}_{1j}(s_r)\Delta\bar{s}/2}{\mathcal{R}_{55}(s)(1+\Delta\bar{s}/2\hat{\chi})}\right)^2 \sigma_{0j}^2, \tag{108}$$

where Eq. (D4) is used,

$$\sigma_{01} = \sigma_{x0}, \qquad \sigma_{02} = \sigma_{x0'}, \qquad \sigma_{06} = \sigma_{H0}, \tag{109}$$

and for j=1,2,6

$$\hat{\mathcal{R}}_{1j}(s_r) = \mathcal{R}_{1j}\,\mathrm{sign}(\kappa_0). \tag{110}$$

With the identity

$$1 - u\frac{\mathcal{R}_{56}(s_r) - \hat{\mathcal{R}}_{16}(s_r)\Delta\bar{s}/2}{\mathcal{R}_{55}(s_r)(1-\Delta\bar{s}/2\hat{\chi}_r)} = \frac{1}{\mathcal{R}_{55}(s_r)(1-\Delta\bar{s}/2\hat{\chi}_r)}, \tag{111}$$

and Eq. (D4), I_A in Eq. (103) becomes

$$I_A(s, \Delta\bar{s}, k) = \frac{ik\,\Pi(s, \Delta\bar{s}, k)\,\exp[ik\,(z_0 - \Delta z_0)]}{\mathcal{R}_{55}(s)(1+\Delta\bar{s}/2\hat{\chi})} \tag{112}$$

for

$$\Delta z_0 = \frac{|R_0|}{\mathcal{R}_{55}(s_r)(1-\Delta\bar{s}/2\hat{\chi}_r)}\left[\frac{\Delta\bar{s}^3}{24} - \left(\bar{z}_1 + \hat{x}_1\frac{\Delta\bar{s}}{2}\right)\right]. \tag{113}$$

For $\Sigma^2(s, \Delta\bar{s})/\sigma_{z0}^2 \ll 1$, using Eqs. (E3) and (D4), we finally get

$$\begin{aligned}(\tilde{F}_H)_1 = {}& \frac{\beta_0^2 N r_e}{\sqrt{2\pi}\gamma_0[\sigma_z(s)]}\int_0^\infty d\Delta\bar{s}\,\frac{\Delta\bar{s}/2}{\mathcal{R}_{55}(s)\sqrt{\Lambda(s,\Delta\bar{s})}(1+\Delta\bar{s}/2\hat{\chi})} \\ & \times\left(\frac{z_0 - \Delta z_0}{\sigma_{z0}^2} - ik_0\right)\exp\left[-\frac{(z_0-\Delta z_0)^2}{2\sigma_{z0}^2} - \frac{k_0^2\Sigma^2(s,\Delta\bar{s})}{2} + ik_0(z_0 - \Delta z_0)\right].\end{aligned} \tag{114}$$

with

$$\sigma_z(s) = |\mathcal{R}_{55}(s)|\sigma_{z0}, \quad \text{and} \quad \bar{\sigma}_z(s) = \frac{\sigma_z(s)}{|R_0|}. \tag{115}$$

Here the range of $\Delta\bar{s}$ is set by $k_0\Delta z_0 \simeq 1$ in Eq. (114). Hence for high frequency k_0 such that

$$O(\Delta\bar{s}) \sim O\left(\frac{\mathcal{R}_{55}(s)}{k_0|R_0|}\right)^{1/3} \ll O[\hat{\chi}(s)], \tag{116}$$

Eq. (114) reduces to the result for a projected bunch.

6.2. Gaussian Initial Longitudinal Distribution

We now look at the effective longitudinal force for an initial Gaussian longitudinal bunch without modulation, namely, $k_0 = 0$ in Eq. (100). Let us now define

$$\alpha(s) = \hat{\xi}(s)[\bar{\sigma}_z(s)]^{1/3}, \tag{117}$$

which, for an initial line bunch with $\sigma_x(s) = \mathcal{R}_{15}(s)\sigma_{z0}$ and $\sigma_z(s)$ in Eq. (115), is related to the parameter in Ref. [9] by

$$\alpha_0(s) = |\alpha(s)| = \frac{\sigma_x(s)}{[\sigma_z(s)^2|R_0|]^{1/3}}. \tag{118}$$

Furthermore, let $\Delta\tilde{s}$ and $\Delta\tilde{z}_0$ be

$$\Delta\tilde{s} = \frac{\Delta\bar{s}}{[\bar{\sigma}_z(s)]^{1/3}} \quad \text{and} \quad \Delta\tilde{z}_0 = \frac{\Delta\tilde{s}^3}{24(1+\alpha\Delta\tilde{s}/2)}. \tag{119}$$

Then for negligible initial intrinsic spreads

$$O\left(\frac{|\mathcal{R}_{5j}(s) - \hat{\mathcal{R}}_{1j}(s)\,\bar{\sigma}_z^{1/3}(s)\,\Delta\tilde{s}/2|\sigma_{0j}}{(1+\alpha\,\Delta\tilde{s}/2)\,\sigma_z(s)}\right) \ll 1 \qquad (j = 1, 2, 6), \tag{120}$$

and for $\mathcal{R}_{55}(s) > 0$, we obtain from Eq. (114) the line bunch result

$$(\tilde{F}_H)_1(\tilde{z}_0, \alpha) \simeq \frac{2Nr_e}{\gamma_0 3^{1/3}\sqrt{2\pi}[\sigma_z(s)]^{4/3}|R_0|^{2/3}}\, I(\tilde{z}_0, \alpha), \tag{121}$$

with

$$I(\tilde{z}_0, \alpha) = \frac{3^{1/3}}{4}\int_0^\infty d\Delta\tilde{s}\frac{\Delta\tilde{s}}{\Lambda_1(\Delta\tilde{s}, \alpha)}(\tilde{z}_0 - \Delta\tilde{z}_0)\exp\left[-\frac{(\tilde{z}_0 - \Delta\tilde{z}_0)^2}{2}\right] \tag{122}$$

for

$$\Lambda_1(\Delta\tilde{s}, \alpha) = \left(1 + \frac{\alpha\Delta\tilde{s}}{2}\right)\sqrt{\left(1 + \frac{\alpha\Delta\tilde{s}}{2}\right)^2 + \frac{(\alpha\Delta\tilde{s})^2}{12}}. \tag{123}$$

The behavior of $I(\tilde{z}_0, \alpha)$ is shown in Fig. 1, which agrees with the simulation results presented in Ref. [9]. Even though our study was carried out for a moderately tilted bunch with $\chi^2 \ll 1$, we note that when $\alpha = 0$ in Eqs. (119)-(121), we recover the results of the effective longitudinal CSR force for a nontilted bunch [8]. As mentioned earlier, the result in Eq. (121) is the contribution from the retardation solution $(z_{r0})_1$. The contribution to $\tilde{F}_H$ from the other retarded solution, $(z_{r0})_2$, is a factor $\alpha^2[\bar{\sigma}_z(s)]^{2/3}$ of that from $(z_{r0})_1$ and is thus neglected here for $O\left([\bar{\sigma}_z(s)]^{1/3}\right) \ll 1$. An example of $\alpha_0(s) = |\alpha(s)|$ vs. s for a typical chicane is shown in Fig. (2), in which $R_0 = 12.2$ m is used as a constant for this plot.

For $\alpha > 0$, the inverse of Eq. (119) is

$$\Delta\tilde{s} = 4\sqrt{\alpha\,\Delta\tilde{z}_0}\cosh\left[\frac{1}{3}\cosh^{-1}\left(\frac{3}{2\alpha\sqrt{\alpha\,\Delta\tilde{z}_0}}\right)\right]. \tag{124}$$

This dependence is shown in Fig. 2. One can see that for a source particle at bunch tail and a test particle at bunch head, with a given separation in the bunch $\Delta\tilde{z}_0$, the interaction between the two particles requires a longer range of pathlength $\Delta\tilde{s}$ when α gets bigger. Let the pathlength at the entrance of the circular orbit be s_1. When $\alpha = 0$, the "steady-state" interaction of the bunch at pathlength s requires $\bar{s} - \bar{s}_1 \geq 2[3\bar{l}_z(s)]^{1/3}$ for $\bar{l}_z(s) = 4\bar{\sigma}_z(s)$. Now for a tilted bunch to reach "steady-state" interaction, we need

$$\bar{s} - \bar{s}_1 \gg O(\Delta\bar{s}) \sim [\bar{\sigma}_z(s)]^{1/3}. \tag{125}$$

The results in Eqs. (114) and (121) are obtained by combining conditions for moderate tilt in Eq. (85), the "steady-state" interaction in Eq. (125), and the approximations in Eq. (C3)

$$\max\left[O(\frac{1}{\gamma_0}), O\left(\frac{\hat{\chi}\hat{x}_1 - \bar{z}_1}{\chi^2}\right)\right] \ll [\bar{\sigma}_z(s)]^{1/3} \ll 1. \tag{126}$$

7. CONCLUSION

In this paper, we studied the dynamics of an ultrarelativistic electron bunch moving on a curved orbit under collective interaction. The equations of motion were obtained from the Hamiltonian of an electron in the bunch, from which the Vlasov equation for the phase space distribution of the electron bunch was derived. With the phase space description chosen so as to explicitly apply the cancellation effect, the integral equation yielded from the Vlasov equation manifests clearly how the phase space distribution is perturbed by the effective CSR forces. After the above formulas for the self-consistent dynamics established, we focused on the analysis of the impact of bunch tilt on the effective longitudinal CSR force, which was carried out under zeroth order approximation for a bunch with initial linear energy chirp. For a simplified initial bunch phase space distribution, we presented the analytical expression of the effective longitudinal CSR force as a function of bunch tilt. These analytical results agree well with Dohlus' simulation results. Many other aspects of the effective CSR forces for a tilted bunch, such as the behavior of the effective transverse CSR force, the behavior of the effective forces around the vicinity of the full compression point, and the effective forces at the entrance and exit of a circular orbit, can be further studied along the approach developed in this paper. A full investigation of the CSR effect in a bunch compression chicane requires a self-consistent numerical simulation, which can also be constructed based on the effective CSR forces as formulated in this paper.

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APPENDIX A: TRANSPORT MATRICES

For $F^{[f]}(X,s)=0$ throughout the beam line, which is often satisfied approximately for a bunch with low charge or long bunch length, Eq. (22) becomes

$$\frac{dX}{ds}=M(s)X, \tag{A1}$$

which describes the linear optical transport of particles. For $X(0)$ being the particle's phase space variable at $s=0$, the solution of Eq. (A1) takes the form

$$X(s)=R(s)X(0), \quad \text{with} \quad R(s)=\begin{pmatrix} R_{11}(s) & R_{12}(s) & 0 & 0 & 0 & R_{16}(s) \\ R_{21}(s) & R_{22}(s) & 0 & 0 & 0 & R_{26}(s) \\ 0 & 0 & R_{33}(s) & R_{34}(s) & 0 & 0 \\ 0 & 0 & R_{43}(s) & R_{44}(s) & 0 & 0 \\ R_{51}(s) & R_{52}(s) & 0 & 0 & 1 & R_{56}(s) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A2}$$

which is the usual linear transport matrix from pathlength 0 to s for a beamline. For a bunch with initial twiss parameters β_{x0}, α_{x0}, β_{y0}, α_{y0}, and an initial energy chirp $\delta_E=uz_0$ ($u<0$ for a bunch under compression) imposed by an upstream RF cavity, one can further define the normalized initial phase space variables $X_0 = (x_0, x_0', y_0, y_0', z_0, \delta_{H0})^T$:

$$X_0(0) = AX(0), \quad \text{for} \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{x0}/\beta_{x0} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{y0}/\beta_{y0} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -u & 1 \end{pmatrix}. \tag{A3}$$

Combining Eqs. (A2) and (A3), we have the transport of phase space vector from the initial normalized phase space to the phase space at s:

$$X(s) = \mathcal{R}(s)X_0(0) \quad \text{for} \quad \mathcal{R}(s) = R(s)A^{-1}, \tag{A4}$$

or,

$$\mathcal{R}(s) = \begin{pmatrix} R_{11}(s) - \frac{\alpha_{x0}}{\beta_{x0}}R_{12}(s) & R_{12}(s) & 0 & 0 & uR_{16}(s) & R_{16}(s) \\ R_{21}(s) - \frac{\alpha_{x0}}{\beta_{x0}}R_{22}(s) & R_{22}(s) & 0 & 0 & uR_{26}(s) & R_{26}(s) \\ 0 & 0 & R_{33}(s) - \frac{\alpha_{y0}}{\beta_{y0}}R_{34}(s) & R_{34}(s) & 0 & 0 \\ 0 & 0 & R_{43}(s) - \frac{\alpha_{y0}}{\beta_{y0}}R_{44}(s) & R_{44}(s) & 0 & 0 \\ R_{51}(s) - \frac{\alpha_{x0}}{\beta_{x0}}R_{52}(s) & R_{52}(s) & 0 & 0 & 1 + uR_{56}(s) & R_{56}(s) \\ 0 & 0 & 0 & 0 & u & 1 \end{pmatrix}. \tag{A5}$$

Substituting Eq. (A4) into Eq. (A1), one finds $\mathcal{R}(s)$ satisfy

$$\frac{d\mathcal{R}(s)}{ds} = M(s)\mathcal{R}(s) \quad \text{and} \quad \frac{d\mathcal{R}^{-1}(s)}{ds} = -\mathcal{R}^{-1}(s)M(s) \tag{A6}$$

where $d(\mathcal{R}\mathcal{R}^{-1})/ds = 0$ is used.

For nonzero effective CSR forces, let us define

$$X_0(s) = \mathcal{R}^{-1}(s)X(s). \tag{A7}$$

From Eqs. (A6) and (22), the change of X_0 is found to be driven by $F^{[f]}$:

$$\frac{dX_0}{ds} = \mathcal{R}^{-1}(s)F^{[f]}[\mathcal{R}(s)X_0, s]. \tag{A8}$$

Note that for vanishing effective CSR forces, $X_0(s)$ is a set of invariant phase space variables, namely, $X_0(s) = X_0(0)$ when $F^{[f]}(X, s) = 0$ for all s.

APPENDIX B: MORE ON THE VLASOV EQUATION

B.1. Small Perturbation on a Stable Distribution

Here we look at the evolution of a small initial perturbation on a stable phase space distribution $f_0(X,s)$

$$f(X,s) = f_0(X,s) + f_1(X,s), \tag{B1}$$

where $f_0(X,s)$ is such that $F^{[f_0]}(X,s)$ is negligible for s along the beamline of interest and thus $f_0(X_\tau,s_\tau) \simeq f_0[\mathcal{R}^{-1}(s_\tau)X_\tau,0]$. A linearized integral equation can be reduced from Eq. (44)

$$f_1(X,s) = f_1^{(0)}(X,s) - \int_0^s ds_\tau \left[F^{[f_1]}(X_\tau,s_\tau)\right]_j \frac{\partial f_0(X_\tau,s_\tau)}{\partial (X_\tau)_j} \tag{B2}$$

with $f_1^{(0)}(X,s) = f_1[\mathcal{R}^{-1}(s)X,0]$. The Fourier component of the perturbed longitudinal distribution at s is [2, 3]

$$g_1(k,s) = \frac{1}{2\pi}\int dX e^{-ikz} f_1(X,s) = g_1^{(0)}(k,s) + \Delta g_1(k,s) \tag{B3}$$

where for $\rho_{0,1}(X_0,0) = f_{0,1}[\mathcal{R}(s)X_0,0]$, and using $dX = dX_\tau = dX_0$,

$$g_1^{(0)}(k,s) = \frac{1}{2\pi}\int dX_0 e^{-ikz(X_0,s)}\rho_1(X_0,0) \tag{B4}$$

and

$$\Delta g_1(k,s) \;=\; \frac{1}{2\pi}\int_0^s ds_\tau \int dX_0\, e^{-ikz(X_0,s)}\,\rho_0(X_0,0)(-ik)R_{5j}(s_\tau\to s)F_j^{[f_1]}[\mathcal{R}(s_\tau)X_0,s_\tau] \tag{B5}$$

with $z(X_0,s) = \mathcal{R}_{5i}(s)\,(X_0)_i$. When only the $j=6$ term is dominant and other terms are negligible, Eq. (B5) reduces to the perturbative version of Eq. (14) in Ref. [3].

B.2. Relation to Measurable Distributions

Because of the unconventional use of δ_H as a dynamical variable in Eqs. (22) and (44), it is necessary to show how the phase space distribution $f(X,s)$ is related to the distributions measured in laboratories.

First, the energy spectrum is often measured at a high dispersion point $s = s_D$, where one measures the horizontal charge density distribution, which is related to $f(X,s)$ by

$$\lambda_x(x,s_D) = \int f(X,s_D)\, dx'dydy'dzd\delta_H. \tag{B6}$$

Note that when the effective CSR forces are negligible, $\lambda_x(x, s_D)$ mainly measures the spread of δ_H at s_D.

Second, the horizontal emittance is often measured in a nondispersive region at $s = s_\epsilon$, which gives the rms area for the following horizontal phase space distribution:

$$\rho_x(x, x', s_\epsilon) = \int f(X, s_\epsilon)\, dydy'dzd\delta_H. \tag{B7}$$

Since according to Eq. (44), $f(X, s_\epsilon)$ in Eq. (B7) is fully determined by the effective CSR forces and the initial distribution $f(X, 0)$, the horizontal emittance is only perturbed by the effective CSR forces.

Next, the longitudinal density distribution is yielded from $f(X, s)$ by

$$\lambda_z(z, s) = \int f(X, s)\, dxdx'dydy'd\delta_H. \tag{B8}$$

Similar to the horizontal emittance, here $\lambda_z(z, s)$ is also perturbed by the effective CSR forces.

However, unlike $\lambda_x(x, s_D)$, $\rho_x(x, x', s_\epsilon)$, and $\lambda_z(z, s)$ which can be determined by $f(X, s)$ with the *canonical* energy offset δ_H as a dynamical variable, the *actual* longitudinal phase space distribution $\rho_z(z, \delta_E, s)$, with δ_E the *kinetic* energy offset, is now related to the scalar potential on the test particles at s following Eq. (32):

$$\rho_z(z, \delta_E, s) = \int f\left(x, x', y, y', z, \delta_E + \tilde{\Phi}(x, y, z, s) - \frac{1}{2\gamma_0^2}, s\right) dxdx'dydy'. \tag{B9}$$

APPENDIX C: SOME DETAILS ON THE RETARDATION SOLUTION

C.1. The Coefficients of the Quadratic Equation $P(\Delta\bar{z}) = 0$

In Sec. 3, the expression $P(x, z, s; X_{r0}, s_r)$ in Eq. (51) is reduced to a quadratic function of $\Delta\bar{z}$ in Eq. (62). One can show that the coefficients in Eq. (62) are

$$\begin{cases} a \;=\; 1 - (\beta_0\xi_r)^2 \\ b \;=\; \Delta\bar{s} + \bar{z}_{r1} + \beta_0^2\hat{\xi}_r\left[(\hat{x} - \hat{x}_{r1} - \hat{\xi}_r\bar{z}) - 2(1 + \hat{x})\sin^2\dfrac{\Delta\bar{s}}{2}\right] \\ c \;\simeq\; \left(\dfrac{\Delta\bar{s}}{\gamma_0}\right)^2 + \dfrac{(\Delta\bar{s})^4}{12} + 2\Delta\bar{s}\bar{z}_{r1} + (\bar{z}_{r1})^2 \\ \qquad -\beta_0^2(\hat{x} - \hat{x}_{r1} - \hat{\xi}_r\bar{z})^2 - [\hat{x} + (1 + \hat{x})(\hat{x}_{r1} + \hat{\xi}_r\bar{z})]\left(2\beta_0\sin\dfrac{\Delta\bar{s}}{2}\right)^2 \end{cases}, \tag{C1}$$

with

$$(\bar{z}_1, \bar{z}_{r1}) = \frac{1}{|R_0|}(z_1, z_{r1}), \quad \text{and} \quad (\hat{x}_1, \hat{x}_{r1}) = \frac{1}{R_0}(x_1, x_{r1}). \tag{C2}$$

Under the approximations

$$\sin(\Delta\bar{s}/2) \simeq \Delta\bar{s}/2, \quad \gamma_0\,\Delta\bar{s} \ll 1, \quad \beta_0^2 \simeq 1, \quad 1+\hat{x} \simeq 1, \tag{C3}$$

the terms in Eq. (C1) can be grouped as

$$b \simeq b_0 + b_1, \quad c \simeq c_0 + c_1, \tag{C4}$$

where b_0 and c_0 are values of b and c when $\hat{x}_{r1} = \bar{z}_{r1} = 0$

$$\begin{cases} b_0 = \Delta\bar{s}\,(1 - \dfrac{\hat{\xi}_r\Delta\bar{s}}{2}) + \hat{\xi}_r(\hat{x} - \hat{\xi}_r\bar{z}) \\ c_0 = \dfrac{(\Delta\bar{s})^4}{12} - (\hat{x} - \hat{\xi}_r\bar{z})^2 - (\hat{x} + \hat{\xi}_r\bar{z})\,(\Delta\bar{s})^2 \end{cases} \tag{C5}$$

and b_1 and c_1 are related to the source particle's initial intrinsic offset via z_{r1} and x_{r1} in Eq. (54)

$$\begin{cases} b_1 = \bar{z}_{r1} - \hat{\xi}_r\hat{x}_{r1}, \\ c_1 = 2\Delta\bar{s}\bar{z}_{r1} + (\bar{z}_{r1})^2 + 2\hat{x}_{r1}\,(\hat{x} - \hat{\xi}_r\bar{z}) - (\hat{x}_{r1})^2 - \hat{x}_{r1}(\Delta\bar{s})^2. \end{cases} \tag{C6}$$

As shown in Eqs. (63)-(65), the roots of $P = 0$ involves

$$\omega = b^2 - ac = \omega_0 + \omega_1 \quad \text{with} \quad \begin{cases} \omega_0 = (b_0)^2 - a\,c_0 \\ \omega_1 = 2b_0 b_1 + (b_1)^2 - a\,c_1 \end{cases} \tag{C7}$$

or

$$\begin{cases} \omega_0 = \Delta\bar{s}^2\left[\left(1 - \dfrac{\hat{\xi}_r\Delta\bar{s}}{2}\right)^2 + \dfrac{(\hat{\xi}_r^2 - 1)}{12}\Delta\bar{s}^2 - 2\hat{\xi}_r^2\hat{x}\right] \\ \qquad\qquad +2\hat{\xi}_r\Delta\bar{s}(\hat{x} - \hat{\xi}_r\bar{z}) + (\hat{x} + \hat{\xi}_r\bar{z})\Delta\bar{s}^2 + (\hat{x} - \hat{\xi}_r\bar{z})^2, \\ \omega_1 = (-2\hat{\xi}_r + \Delta\bar{s})\Delta\bar{s}(\hat{x}_{r1} - \hat{\xi}_r\bar{z}_{r1}) + (\hat{x}_{r1} - \hat{\xi}_r\bar{z}_{r1})^2 - 2(\hat{x} - \hat{\xi}_r\bar{z})(\hat{x}_{r1} - \hat{\xi}_r\bar{z}_{r1}). \end{cases} \tag{C8}$$

The factor $(\omega_0 + \omega_1)\hat{\chi}^2$ in Eq. (69) then yields:

$$\begin{aligned} (\omega_0 + \omega_1)\hat{\chi}_r^2 = \Delta\bar{s}^2&\left[\left(\hat{\chi}_r - \frac{\Delta\bar{s}}{2}\right)^2 + \frac{1 - \hat{\chi}_r^2}{12}\Delta\bar{s}^2 - 2\hat{x}\right] + 2\Delta\bar{s}(\hat{\chi}_r\hat{x} - \bar{z}) + \hat{\chi}_r(\hat{\chi}_r\hat{x} + \bar{z})\Delta\bar{s}^2 \\ &+(\hat{\chi}_r\hat{x} - \bar{z})^2 - 2(\hat{\chi}_r\hat{x}_{r1} - \bar{z}_{r1})\left[\Delta\bar{s}\left(1 - \frac{\hat{\chi}_r\Delta\bar{s}}{2}\right) + (\hat{\chi}_r\hat{x} - \bar{z})\right] + (\hat{\chi}_r\hat{x}_{r1} - \bar{z}_{r1})^2. \end{aligned} \tag{C9}$$

C.2. Existence of Solutions: $\theta(b^2 - ac) = 1$

The solutions to $P = 0$, which are $\Delta\bar{z}^{(\pm)}$ in Eq. (64) or $\bar{z}_{r0}^{(\pm)}$ in Eqs. (67) or (69), exist only when $\omega = b^2 - ac \geq 0$, or $\theta(b^2 - ac) = 1$. When X and s for a test particle are given, and the pathlength s_r for the source particle is given, this condition defines the range of Z_{r0}.

In the case when $\hat{\chi}^2 \ll 1$, and when σ_{0j} in Eq. (109) and $\Delta\bar{s}$ satisfy

$$O\left(\frac{[\hat{\chi}\mathcal{R}_{1j}(s) - \mathcal{R}_{5j}(s)]\sigma_{0j}}{|R_0|\hat{\chi}^2}\right) \ll \Delta\bar{s}, \tag{C10}$$

we have for $\Delta\bar{s} > 0$

$$\Delta\bar{s}\left[\left(\hat{\chi}_r - \frac{\Delta\bar{s}}{2}\right)^2 + \frac{\Delta\bar{s}^2}{12}\right] \gg O(\hat{\chi}\hat{x}_1 - \bar{z}_1) \tag{C11}$$

for the full range of bunch intrinsic distribution $(Z_{r0})_j = (-5\sigma_{0j}, 5\sigma_{0j})$ with $j = 1, 2$ and 6. Consequently from Eq. (C9) one gets

$$(b^2 - ac)\chi_r^2 \simeq \Delta\bar{s}^2\left[\left(\hat{\chi}_r - \frac{\Delta\bar{s}}{2}\right)^2 + \frac{\Delta\bar{s}^2}{12}\right] > 0, \tag{C12}$$

or $\theta(b^2 - ac) = 1$. In the situation when σ_{0j} and $\Delta\bar{s}$ are such that Eq. (C10) is not satisfied, one needs to solve the range of Z_{r0} using $(b^2 - ac)\chi_r^2 \geq 0$ in Eq. (C9).

C.3. Selection of Retardation Solutions: $\theta(Q^{(\pm)}) = 1$

Given a test particle at (x, z, s), for a source particle emitting fields at the pathlength s_r, or for fixed $\Delta\bar{s} = (s - s_r)/|R_0|$, we have identified in Eqs. (67) and (69) the longitudinal position $\bar{z}_{r0}^{(\pm)}$ of the source particle in the initial phase space distribution, with $\bar{z}_{r0}^{(\pm)}$ either the "advanced" or "retarded" solution. Here we need to find the range of ξ_r for $\bar{z}_{r0}^{(-)}$ (or $\bar{z}_{r0}^{(-)}$) to be the "retarded" solution, i.e., $\theta(Q^{(-)}) = 1$ (or $\theta(Q^{(+)}) = 1$), provided $\theta(b^2 - ac) = 1$.

First, for mild tilt with $O(\xi_r)$ ranging from 0 to $O(1)$, using $O(\bar{z}_{r1}) \ll \Delta\bar{s} < 1$, we have

$$Q^{(-)} \simeq \Delta\bar{s} - \frac{b - \sqrt{b^2 - ac}}{a} \simeq \begin{cases} \Delta\bar{s} & (\Delta\bar{s} > 0) \\ \dfrac{-(1 - \hat{\xi}_r\Delta\bar{s}) - \hat{\xi}_r^2}{1 - \hat{\xi}_r^2}\Delta\bar{s} & (\Delta\bar{s} < 0) \end{cases}, \tag{C13}$$

and

$$Q^{(+)} \simeq \Delta\bar{s} - \frac{b + \sqrt{b^2 - ac}}{a} \simeq \begin{cases} \dfrac{-(1 - \hat{\xi}_r\Delta\bar{s}) - \hat{\xi}_r^2}{1 - \hat{\xi}_r^2}\Delta\bar{s} & (\Delta\bar{s} > 0) \\ \Delta\bar{s} & (\Delta\bar{s} < 0) \end{cases}, \tag{C14}$$

which gives

$$\theta(Q^{(-)}) = \begin{cases} 1 & (\Delta\bar{s} > 0,\ \text{or } \Delta\bar{s} < 0 \text{ and } \hat{\xi}_r^2 < 1) \\ 0 & (\Delta\bar{s} < 0 \text{ and } \hat{\xi}_r^2 > 1) \end{cases} \tag{C15}$$

and

$$\theta(Q^{(+)}) = \begin{cases} 1 & (\Delta\bar{s} > 0 \text{ and } \hat{\xi}_r^2 > 1) \\ 0 & (\Delta\bar{s} < 0,\ \text{or } \Delta\bar{s} > 0 \text{ and } \hat{\xi}_r^2 < 1) \end{cases}. \tag{C16}$$

Next, for strong tilt, $\hat{\chi}_r^2 \ll 1$, with $O(\bar{z}_{r1}) \ll \Delta\bar{s} < 1$, we have

$$Q^{(\pm)} \simeq \frac{\Delta\bar{s}\left(1 - \hat{\chi}_r \frac{\Delta\bar{s}}{2}\right) \pm \hat{\chi}_r|\Delta\bar{s}|\sqrt{\left(\hat{\chi}_r - \frac{\Delta\bar{s}}{2}\right)^2 + \frac{(1-\hat{\chi}_r^2)\Delta\bar{s}^2}{12}}}{1 - \hat{\chi}_r^2}, \tag{C17}$$

and one can show

$$\theta(Q^{(\pm)}) = \begin{cases} 1 & (\Delta\bar{s} > 0) \\ 0 & (\Delta\bar{s} < 0). \end{cases} \tag{C18}$$

As discussed in Sec. 4, for $\xi_r = 0$, the source particle $\bar{z}_{r0}^{(+)}$ is ahead of the test particle $z_0 = z/\mathcal{R}_{55}(s)$ in the initial bunch, while the source particle $\bar{z}_{r0}^{(-)}$ is behind the test particle z_0. However, it is interesting to note that when $\xi_r^2 > 1$, the test particle receives fields from the source particles $\bar{z}_{r0}^{(+)}$ and $\bar{z}_{r0}^{(-)}$, both generated from pathlength s_r behind s, or $s_r < s$.

APPENDIX D: MORE ON THE RETARDED TILTING FACTOR

From the definition of $\chi(s_r)$ and $R_{56}'(s) = -R_{16}(s)/R_0(s)$, one gets

$$\chi(s_r) = \frac{1 + uR_{56}(s - \Delta s)}{uR_{16}(s - \Delta s)} \simeq \frac{R_{16}(s)}{R_{16}(s - \Delta s)}\left[\chi(s) + \frac{\Delta s}{R_0(s)} - \frac{R_{26}(s)}{2R_{16}(s)R_0(s)}(\Delta s)^2\right], \tag{D1}$$

where

$$\frac{R_{16}(s - \Delta s)}{R_{16}(s)} \simeq 1 - \left(\frac{R_{26}(s)|R_0(s)|}{R_{16}(s)}\right)\Delta\bar{s} + \frac{R_0(s)}{2R_{16}(s)}(\Delta\bar{s})^2. \tag{D2}$$

Eq. (D1) further yields

$$\hat{\chi}(s_r) - \frac{\Delta\bar{s}}{2} \simeq \frac{R_{16}(s)}{R_{16}(s_r)}\left(\hat{\chi}(s) + \frac{\Delta\bar{s}}{2} - \frac{R_0(s)}{4R_{16}(s)}(\Delta\bar{s})^3\right). \tag{D3}$$

For $O[R_0\Delta\bar{s}^2/R_{16}(s)] \ll 1$, Eq. (D3) becomes

$$\hat{R}_{16}(s_r)\left(\hat{\chi}_r - \frac{\Delta\bar{s}}{2}\right) = \mathcal{R}_{55}(s_r)\left(1 - \frac{\Delta\bar{s}}{2\hat{\chi}_r}\right) \simeq \mathcal{R}_{55}(s)\left(1 + \frac{\Delta\bar{s}}{2\hat{\chi}(s)}\right), \tag{D4}$$

and it can be shown that

$$|uR_{16}(s_r)|\sqrt{\left(\hat{\chi}_r - \frac{\Delta\bar{s}}{2}\right)^2 + \frac{(\Delta\bar{s})^2}{12}} \simeq |\mathcal{R}_{55}(s)|\sqrt{\left(1 + \frac{\Delta\bar{s}}{2\hat{\chi}(s)}\right)^2 + \frac{(\Delta\bar{s})^2}{12\chi^2(s)}}. \tag{D5}$$

APPENDIX E: SOME INTEGRALS

We start from the integral

$$\begin{aligned} I_0 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,\exp\left[-\frac{1}{2}(k-k_0)^2\sigma_{z0}^2 - \frac{1}{2}k^2\Sigma^2 + i\,k(z_0-\Delta z_0)\right] \\ &= \frac{1}{\sqrt{2\pi}\sigma_{z0}\sqrt{1+\mu_0}}\exp\left[-\frac{1}{2(1+\mu_0)}\left(k_0^2\Sigma^2 + \frac{(z_0-\Delta z_0)^2}{\sigma_{z0}^2}\right) + \frac{ik_0(z_0-\Delta z_0)}{1+\mu_0}\right] \end{aligned} \tag{E1}$$

for

$$\mu_0 = \Sigma^2/\sigma_{z0}^2. \tag{E2}$$

For $\mu_0 \ll 1$, Eq. (E1 yields

$$\begin{aligned} \frac{\partial I_0}{\partial z_0} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,(ik)\exp\left[-\frac{1}{2}(k-k_0)^2\sigma_{z0}^2 - \frac{1}{2}k^2\Sigma^2 + i\,k(z_0-\Delta z_0)\right] \\ &\simeq \frac{1}{\sqrt{2\pi}\sigma_{z0}}\left(-\frac{z_0-\Delta z_0}{\sigma_{z0}^2} + ik_0\right)\exp\left[-\frac{k_0^2\Sigma^2}{2} - \frac{(z_0-\Delta z_0)^2}{2\sigma_{z0}^2} + ik_0(z_0-\Delta z_0)\right]. \end{aligned} \tag{E3}$$

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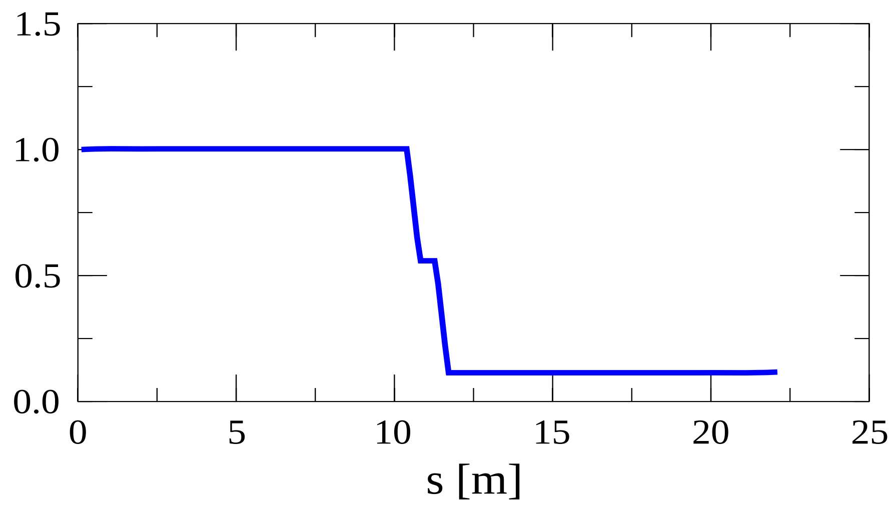

FIG. 1: Compression factor, $\mathcal{R}_{55}(s) = 1 + uR_{56}(s)$, vs. s for $u = -40$ m^{-1}.

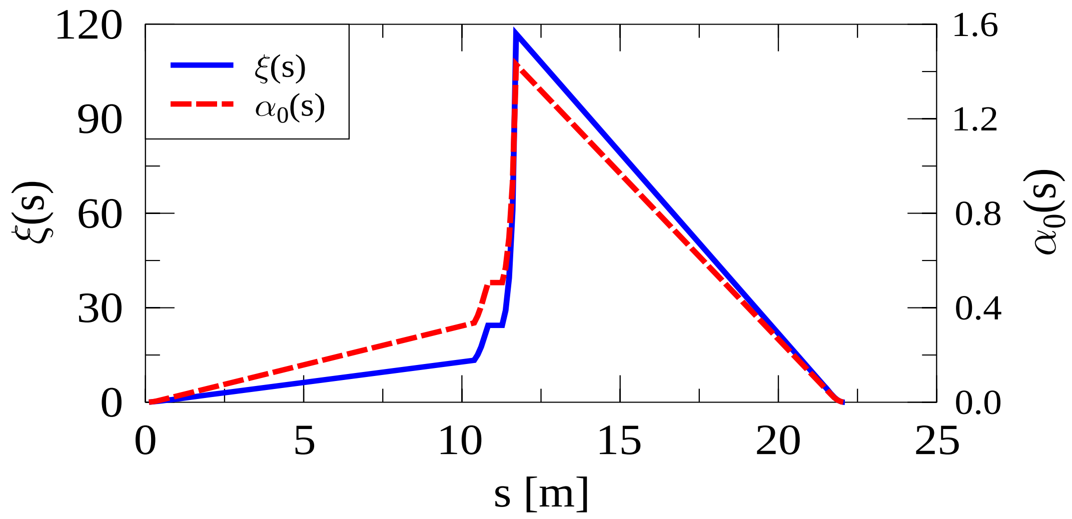

FIG. 2: Tilt factor or x-z slope, $\xi(s)$ (defined in Eq. (59)), and parameter $\alpha_0(s)$ in Eq. (118) for $u = -40$ m^{-1} (assuming positive curvature for the center two dipoles).

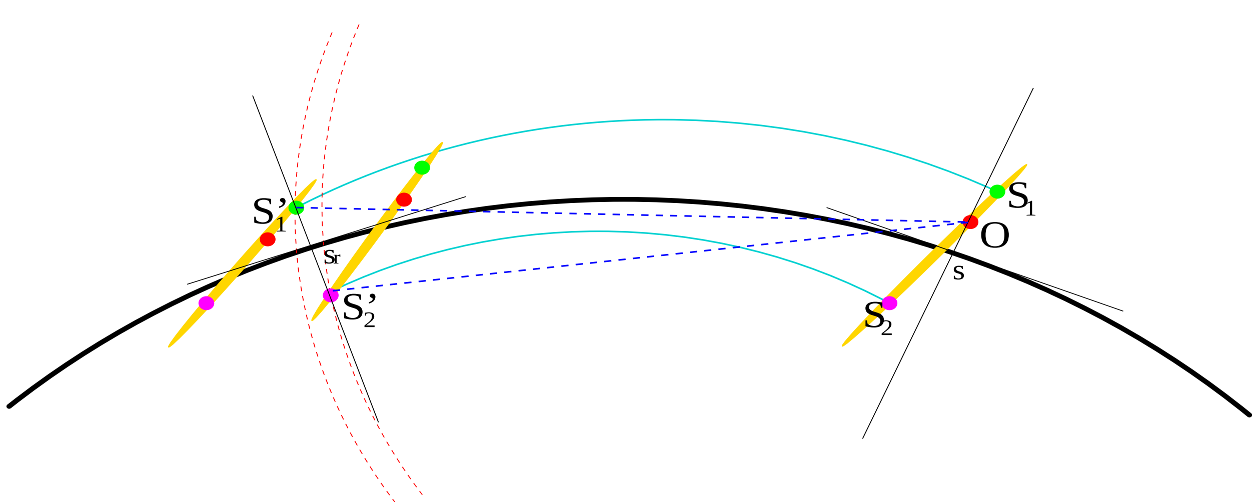

FIG. 3: Illustration of the interaction of the source particles S_1' and S_2' at pathlength s_r with the test particle O at pathlength s for a tilted bunch. The retardation requires $\widehat{\mathrm{S}_1'\mathrm{S}_1} = (v_1/c)\,\overline{\mathrm{S}_1'\mathrm{O}}$ and $\widehat{\mathrm{S}_2'\mathrm{S}_2} = (v_2/c)\,\overline{\mathrm{S}_2'\mathrm{O}}$ (may not seen from drawing which is not to scale). The source particles S_1' and S_2' are at the interceptions of $s = s_r$ with the the past lightcones (dashed red arcs) of the test particle O at (x, z, s), with retarded velocity v_1 and v_2 respectively.

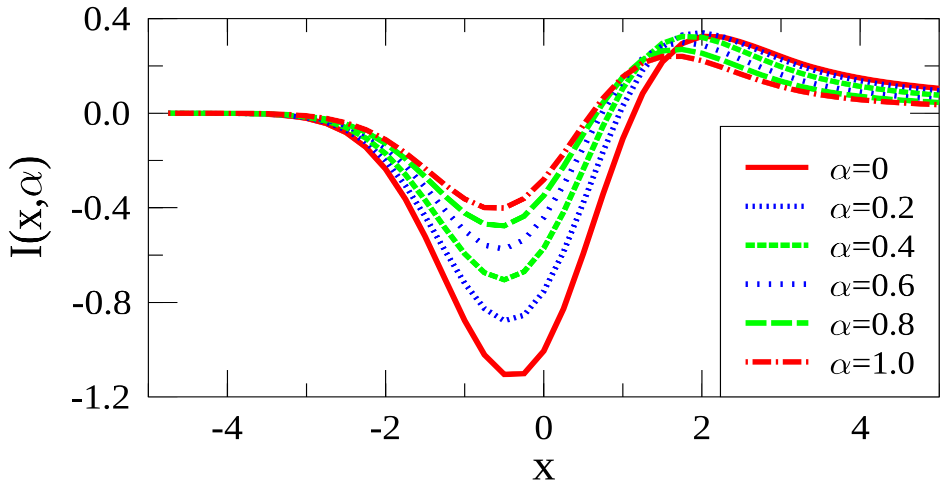

FIG. 4: $I(x, \alpha)$ vs. x for various α given by Eq. (122).

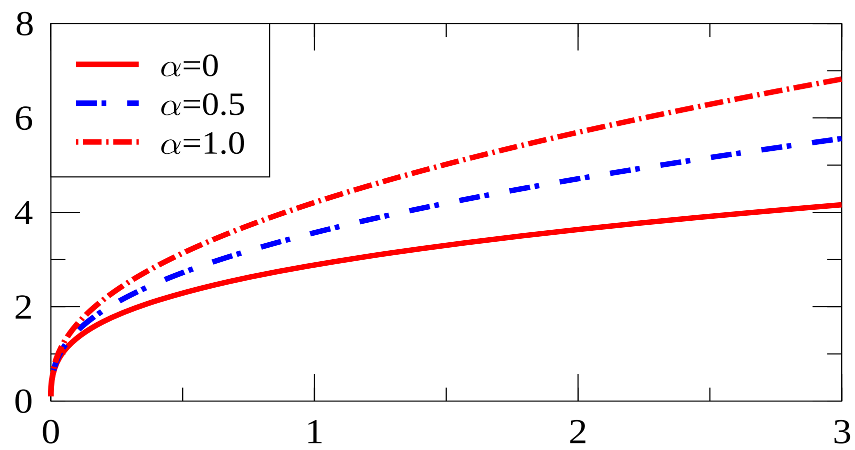

FIG. 5: $\Delta\tilde{s}$ vs. $\Delta\tilde{z}_0$ for various α given by Eq. (124).