Field Error Tolerances – Quadrupole/Dipole Specs

Alex Bogacz and Mike Tiefenback

1. Field Error Tolerances – Analytic Approach

A single point mismatch due to a focusing error can be measured by the following Courant-Snyder invariant change:

$$
\begin{aligned}
\varepsilon' &= \beta(\theta+\delta\theta)^2 + 2\alpha(\theta+\delta\theta)x + \gamma x^2 \\
&= \varepsilon + 2\left(\beta\theta+\alpha x\right)\delta\theta + \beta\delta\theta^2 \quad ,
\end{aligned}
$$

where (1)

$$x = \sqrt{\varepsilon\beta}\sin\mu, \quad \theta = \sqrt{\frac{\varepsilon}{\beta}}\sin\mu\left(\cos\mu - \alpha\sin\mu\right)$$

Each source of field error (magnet) contributes the following Courant-Snyder variation:

$$\delta\varepsilon = \varepsilon' - \varepsilon = 2\sqrt{\varepsilon\beta}\cos\mu\,\delta\theta + \beta\delta\theta^2 \quad , \tag{2}$$

Therefore, a focusing 'point' error perturbs the betatron motion leading to the Courant-Snyder invariant change as illustrated heuristically in Figure 1.

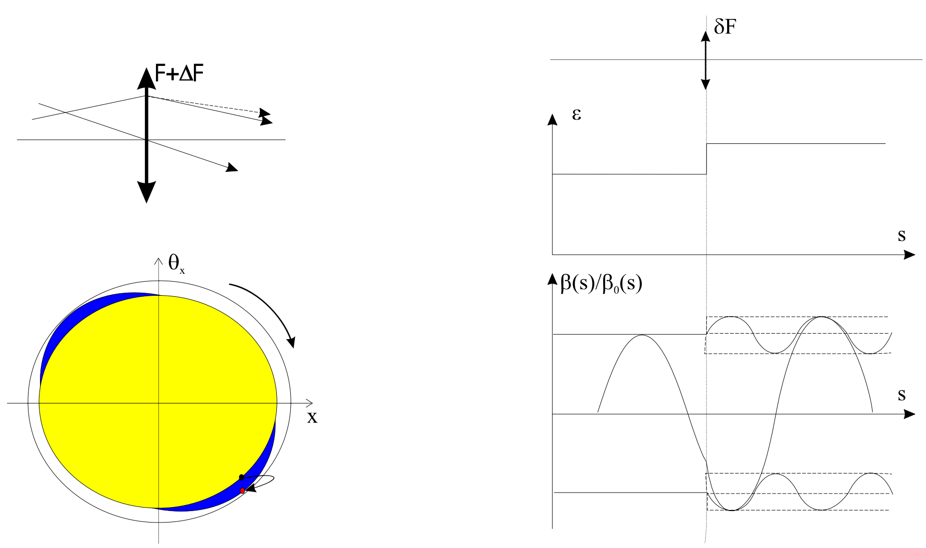

Figure 1 Courant-Snyder invariant change due to perturbed betatron motion

We also assume that the beam trajectory is perturbed with respect to the design orbit and the beam centroid undergoes a betatron oscillation with amplitude X_0. The superposition of both the coherent and incoherent betatron motions is illustrated in Figure 2.

The angular deviation, $\delta\theta$, in Eq.(2) results from the multipole content of each magnet and depends on the individual particle trajectories as they sample the nonlinear part of the magnetic field. Figure 2 illustrates weighted sampling of the nonlinear potential across the ensemble of particles. This can be expressed by the following approximation

$$\delta\theta = \frac{\int B^{grad} dl}{B\rho} = \sum_{m=1} \delta\phi_m x^m \approx \sum_{m=1} \frac{\partial}{\partial x}\left(\delta\phi_m X^m\right)\Delta x = \sum_{m=1} \delta\phi_m m X^{m-1}\Delta x \approx \sum_{m=1} m\frac{\Delta x}{X_0}\left(\delta\phi_m X^m\right), \tag{3}$$

where $\delta\phi_m = \int \delta k_m dl,\quad X=X_0 \sin\mu \quad \text{and} \quad \Delta x=2a \quad a=4\sqrt{\beta\varepsilon}$

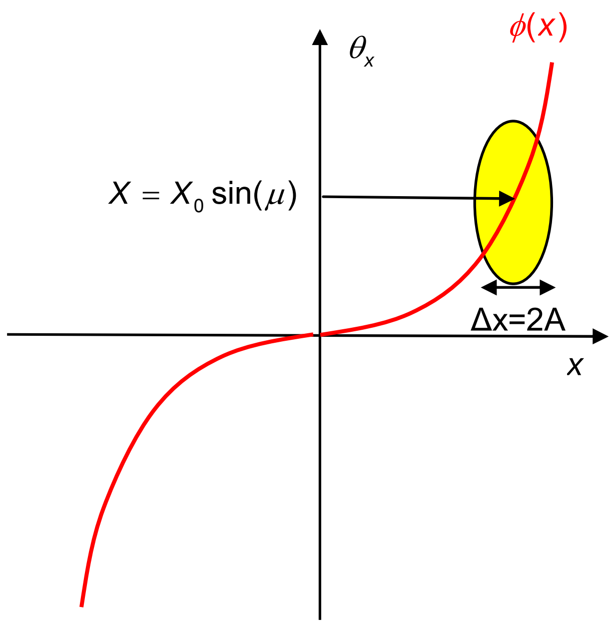

Figure 2 Weighted sampling of the nonlinear potential across the ensemble of particles in a bunch which undergoes coherent betatron oscillation with amplitude X_0.

Substituting ensemble averaged $\delta\theta$ given by Eq.(3) into Eq.(2) yields the following expression for the emittance variation:

$$\delta\varepsilon = 4\varepsilon\beta \sum_{m=1} m X_0^{\,m-1}\delta\phi_m \cos\mu \ \sin^m\mu \quad + 4\varepsilon\beta^2\left(\sum_{m=1} m X_0^{\,m-1}\delta\phi_m \ \sin^m\mu\right)^2, \tag{4}$$

here m = 2 (quadrupole), m =3 (sextupole), m =4 (octupole) etc.

Considering N uncorrelated sources of field errors, a cumulative mismatch along the lattice can be written as follows

$$\varepsilon_N = \varepsilon \prod_{n=1}^{N} \left(1 + 4\beta \sum_{m=1} m {X_0}^{m-1} \delta\phi_m \cos\mu \ \sin^m \mu \quad + 4\beta^2 \left(\sum_{m=1} m {X_0}^{m-1} \delta\phi_m \ \sin^m \mu \right)^2 \right) \quad , \tag{5}$$

Using Eq. (5) one can calculate the standard deviation of the Courant-Snyder invariant. This yields the following expression:

$$\frac{\sigma_\varepsilon}{\varepsilon} = \frac{\sqrt{\left\langle \delta\varepsilon^2 \right\rangle - \left\langle \delta\varepsilon \right\rangle^2}}{\varepsilon} =$$

$$\sqrt{\sum_{i=1}^{N} \left[2\beta_i \sum_{m=1} m {X_0}^{m-1} \delta\phi_m \left\langle \cos\mu \ \sin^m \mu \right\rangle \quad + \beta_i^2 \left\langle \left(\sum_{m=1} m {X_0}^{m-1} \delta\phi_m \ \sin^m \mu \right)^2 \right\rangle \right]} \tag{6}$$

For a weakly focusing lattice (uniform beta modulation) the following averaging (over the betatron phase) can by applied:

$$\langle \ldots \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\mu \ldots \tag{7}$$

Applying some useful integrals:

$$\left\langle \cos\mu \ \sin^m \mu \right\rangle = 0 \quad ,$$

$$\left\langle \sin^m \mu \right\rangle = \frac{m-1}{m} \left\langle \sin^{m-2} \mu \right\rangle = \begin{cases} 0 & \text{m odd} \\ \dfrac{(m-1)!!}{m!!} & \text{m even} \end{cases} \tag{8}$$

and including only the first few multipoles will reduce coherent contribution to the C-S variance, Eq.(6), as follows:

$$\frac{\sigma_\varepsilon}{\varepsilon} = \sqrt{\sum_{i=1}^{N} \left\{ \beta_i^2 \left[\delta\phi_1^2 \underset{\downarrow \atop \frac{1}{2}}{\left\langle \sin^2 \mu \right\rangle} + 2\varepsilon\beta_i \left((2\times2)\delta\phi_2^2 + 2(1\times3)\delta\phi_1\delta\phi_3 \right) \underset{\downarrow \atop \frac{1}{2}\,\frac{3}{4}}{\left\langle \sin^4 \mu \right\rangle} + \varepsilon\beta_i X_0 \left((3\times3)\delta\phi_3^2 + 2(1\times5)\delta\phi_1\delta\phi_5 + 2(2\times4)\delta\phi_2\delta\phi_4 \right) \underset{\downarrow \atop \frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}}{\left\langle \sin^6 \mu \right\rangle} + \ldots \right] \right\}}$$

(9)

Introducing the beam radius at a given magnet as:

$$a_i = 4\sqrt{\varepsilon\beta_i} \tag{10}$$

one can define the overall beam radius at for a given type of magnets as follows:

$$a = 4\sqrt{\varepsilon\overline{\beta}} \tag{11}$$

Assuming the same multipole content for all magnets in the class further simplifies Eq.(9):

$$\frac{\sigma_\varepsilon}{\varepsilon} = \sqrt{\sum_{i=1}^{N}\frac{1}{2}\beta_i^{\,2}} \times \Delta\Phi \tag{12}$$

The first factor purely depends on the beamline optics (focusing), while the second one describes field errors and magnet nonlinearities. The second factor can be re-written in terms of the multipole gradients as follows;

$$\begin{aligned}\Delta\Phi^2 = {} & \delta\phi_1^{\,2} + \frac{3}{2}\frac{1}{2^2}a^2\left[(2\times3)\delta\phi_2^{\,2} + 2(1\times3)\delta\phi_1\delta\phi_3\right] + \frac{5}{2}\frac{1}{2^4}a^2X_0^2\left[(3\times3)\delta\phi_3^{\,2} + 2(1\times5)\delta\phi_1\delta\phi_5 + 2(2\times4)\delta\phi_2\delta\phi_4\right] \\ & + \frac{7}{2}\frac{1}{2^6}a^2X_0^4\left[(4\times4)\delta\phi_4^{\,2} + 2(1\times7)\delta\phi_1\delta\phi_7 + 2(2\times6)\delta\phi_2\delta\phi_6 + 2(3\times5)\delta\phi_3\delta\phi_5\right] \\ & + \frac{9}{2}\frac{1}{2^8}a^2X_0^6\left[(5\times5)\delta\phi_5^{\,2} + 2(1\times9)\delta\phi_1\delta\phi_9 + 2(2\times8)\delta\phi_2\delta\phi_8 + 2(3\times7)\delta\phi_3\delta\phi_7 + 2(4\times6)\delta\phi_4\delta\phi_6\right] + \ldots.\end{aligned}$$

here
$$\phi_n = \frac{\int G_n dl}{B\rho} = \int k_n dl \tag{13}$$

is the integrated multipole moment in the geometric units, where

$$G_m = \frac{1}{r_0^{\,m}}\mathrm{B}_{m+1}\ \left[kGauss\ cm^{-m}\right] \qquad\qquad k_n = \frac{G_n}{B\rho}\ \left[cm^{-(n+1)}\right]$$

Here, one uses multipole expansion coefficients of the azimuthal magnetic field, B_θ, given by the following Fourier series representation in polar coordinates (at a given point along the trajectory):

$$B_\theta(r,\theta) = \sum_{m=2}\left(\frac{r}{r_0}\right)^{m\text{-}1}\left(B_m\cos m\theta + A_m\sin m\theta\right) \tag{14}$$

Finally, one can use Eq(12) to set limits on magnet field quality in terms of ad hoc assigned allowances for betatron mismatch and emittance dilution through a given beamline.

The linear errors, m =1, cause the betatron mismatch – invariant ellipse distortion from the design ellipse without changing its area – no emittance increase. By design, one can tolerate some level of Arc-to-Arc betatron mismatch due to the focusing errors, $\delta\phi_1$ (quad gradient errors and dipole body gradient) to be compensated by the dedicated matching quads

$$\left(\frac{\sigma_\varepsilon}{\varepsilon}\right)_{mis} = \sqrt{\frac{1}{2}\sum_{n=1}^{N}\left(\beta_n\delta\phi_1\right)^2} = \sqrt{\frac{1}{2}\Delta\phi_1^2\sum_{n=1}^{N}\left(\beta_n\right)_{quad}^2 + \frac{1}{2}\delta\phi_1^2\sum_{n=1}^{N}\left(\beta_n\right)_{dipole}^2} \tag{15}$$

The higher, m > 1, multipoles will contribute to the emittance dilution – 'limited' by design via a separate allowance per each segment (Arc, linac). Here one assumes the following multipole content for the dipoles and quads:

- Quads: sextupole (m = 2), octupole (m = 3), duodecapole (m = 5) and isacopole (m = 9)
- Dipoles: sextupole (m = 2) and decapole (m = 4)

$$\left(\frac{\sigma_\varepsilon}{\varepsilon}\right)_{dil} = \sqrt{\frac{1}{2}\sum_{n=1}^{N}\left(\beta_n\delta\phi\right)^2} = \sqrt{\frac{1}{2}\Delta\phi_{quad}^2\sum_{n=1}^{N}\left(\beta_n\right)_{quad}^2 + \frac{1}{2}\Delta\phi_{dipole}^2\sum_{n=1}^{N}\left(\beta_n\right)_{dipole}^2}$$

where (16)

$$\Delta\phi_{quad} = a\sqrt{\frac{3}{2}\frac{1}{2^2}(2\times 2)\delta\phi_2^{\ 2} + \frac{5}{2}\frac{1}{2^4}X_0^{\ 2}\left[(3\times 3)\delta\phi_3^{\ 2} + 2(1\times 5)\delta\phi_1\delta\phi_5\right] + \frac{9}{2}\frac{1}{2^8}X_0^{\ 6}\left[(5\times 5)\delta\phi_5^{\ 2} + 2(1\times 9)\delta\phi_1\delta\phi_9\right]}$$

$$\Delta\phi_{dipole} = a\sqrt{\frac{3}{2}\frac{1}{2^2}(2\times 2)\delta\phi_2^{\ 2} + 2\frac{5}{2}\frac{1}{2^4}X_0^{\ 2}(2\times 4)\delta\phi_2\delta\phi_4 + \ldots}$$

One can use the above analytic formalism to set the magnet error tolerances for specific groups (types) of magnets (dipoles and quads) within each lattice segment (Arcs, linacs). For each group of magnets within each segment one needs to evaluate:

$$\sqrt{\frac{1}{2}\sum_{n=1}^{N}\beta_n^{\ 2}} \tag{17}$$

2. Magnet Tolerances for 12 GeV CEBAF Lattice

Here some limits on tolerable errors and magnet field quality will be set in terms of the following allowances for betatron mismatch (10% increase) and emittance dilution (10% increase) through each piece of beamline (Arc, linac).

The linear errors, focusing (m =1), cause the betatron mismatch (invariant ellipse distortion from the design) without emittance increase. The sources of the betatron mismatch come from the quad gradient errors and the dipole body gradients (to be compensated by the dedicated matching quads)

The higher, m > 1, multipoles will contribute to the emittance dilution – 'limited' by design via a separate allowance.

Both the linear error and the higher multipole tolerances will be addressed in this study separately.

Using specific lattices for 12 GeV CEBAF one can generate beta dependent sums given by Eq.(17) for individual machine segments: 10 Arcs and 2 5-pass linacs (total of 15 entities)

<u>2.1.Linacs</u>

Using multi-pass optics for both North (NL) and South (SL) linacs, illustrated in Figure 2, one can evaluate the expression given by Eq.(17) for each pass. Tables 1 and 2 summarize these values for NL and SL respectively. Then the focusing error tolerances (10% mismatch) for the linac quads were evaluated via Eq.(12); the resulting gradient errors are collected in Tables 1 and 2, as well.

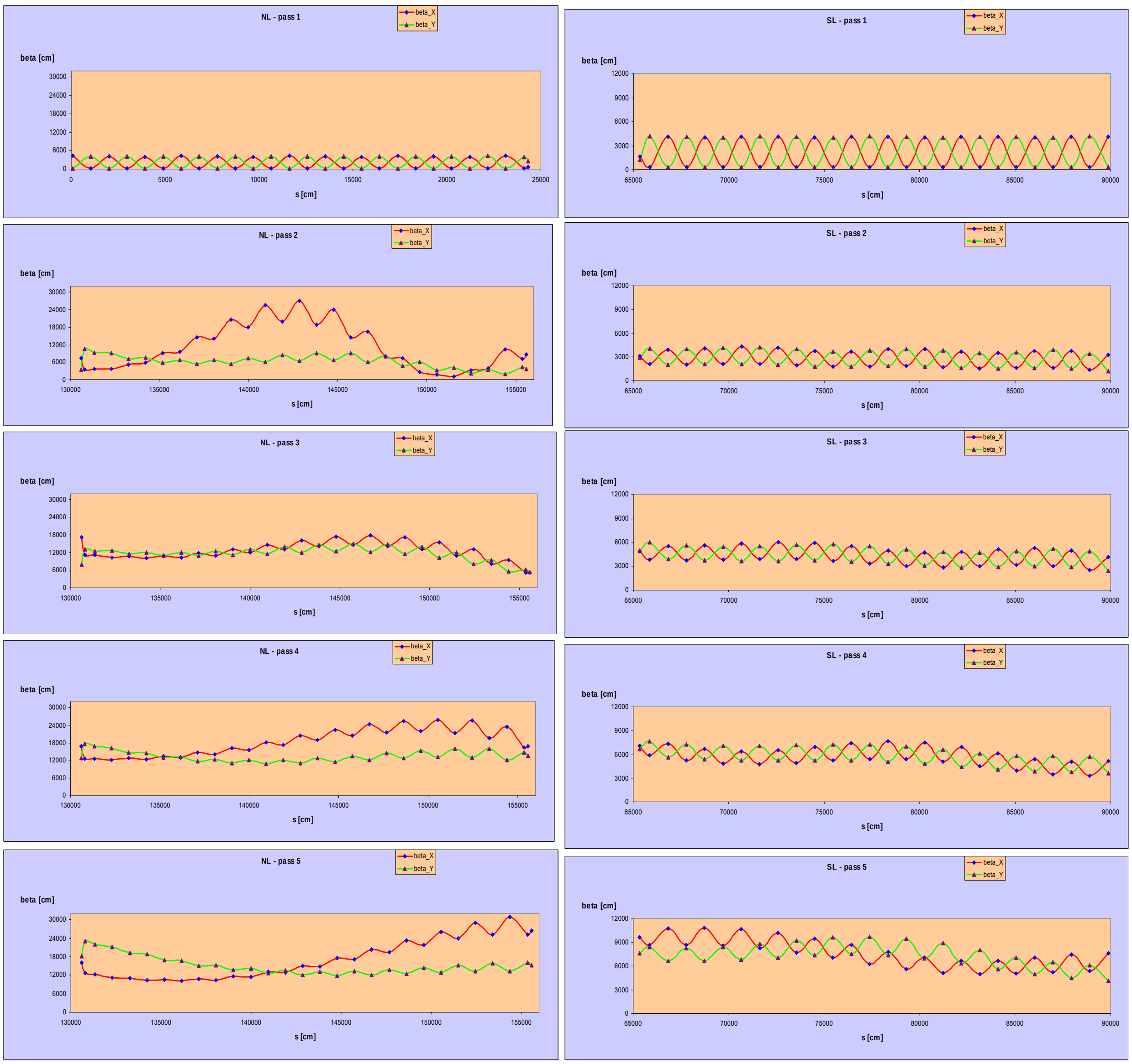

Figure 2 Beta functions for multi-pass linacs: NL (left) and SL(right)

NL pass	$\sqrt{\sum_{n=1}^{N} \beta_n^2}$ $[cm]$	$\Delta\phi_1 \left[cm^{-1}\right]$	$\phi_1^{\max}\left[cm^{-1}\right]$	$\Delta\phi_1 / \phi_1^{\max}$
1	10653	6.7E-06	1.8E-03	3.7E-03
2	50244	1.4E-06	6.5E-04	2.2E-03
3	48809	1.4E-06	3.9E-04	3.7E-03
4	71121	9.9E-07	2.8E-04	3.5E-03
5	70265	1.0E-06	2.2E-04	4.5E-03
6	72548	9.7E-07	1.8E-04	5.4E-03

Table 1 NL quads focusing error tolerances

SL pass	$\sqrt{\sum_{n=1}^{N} \beta_n^2}$ $[cm]$	$\Delta\phi_1 \left[cm^{-1}\right]$	$\phi_1^{\max}\left[cm^{-1}\right]$	$\Delta\phi_1 / \phi_1^{\max}$
1	10551	6.7E-06	1.9E-03	3.6E-03
2	11115	6.4E-06	9.5E-04	6.7E-03
3	16379	4.3E-06	6.4E-04	6.7E-03
4	21696	3.3E-06	4.8E-04	6.7E-03
5	28860	2.5E-06	3.9E-04	6.3E-03

Table 2 SL quads focusing error tolerances

Here the highlighted (yellow) is the most stringent tolerance, which occurs for NL 2-nd pass and SL 1-st pass. This focusing error tolerances are chosen for NL and SL quads respectively

2.2 Arcs

Using Arc 1-10 optics, illustrated in Figure 3, one can evaluate Eq.(17) for each pas. Table 3 (quads) and Table 4 (dipoles) summarize focusing error tolerances (10% mismatch) and field quality specs – higher multipoles (10% emittance dilution) for various groups of quads and dipoles in the corresponding Arcs 1-10, NL and SL are evaluated via Eq.(13). The values of multipoles are calculated in the extreme case – a given order (m) multipole by itself exhausts the emittance dilution allowance of 10%.

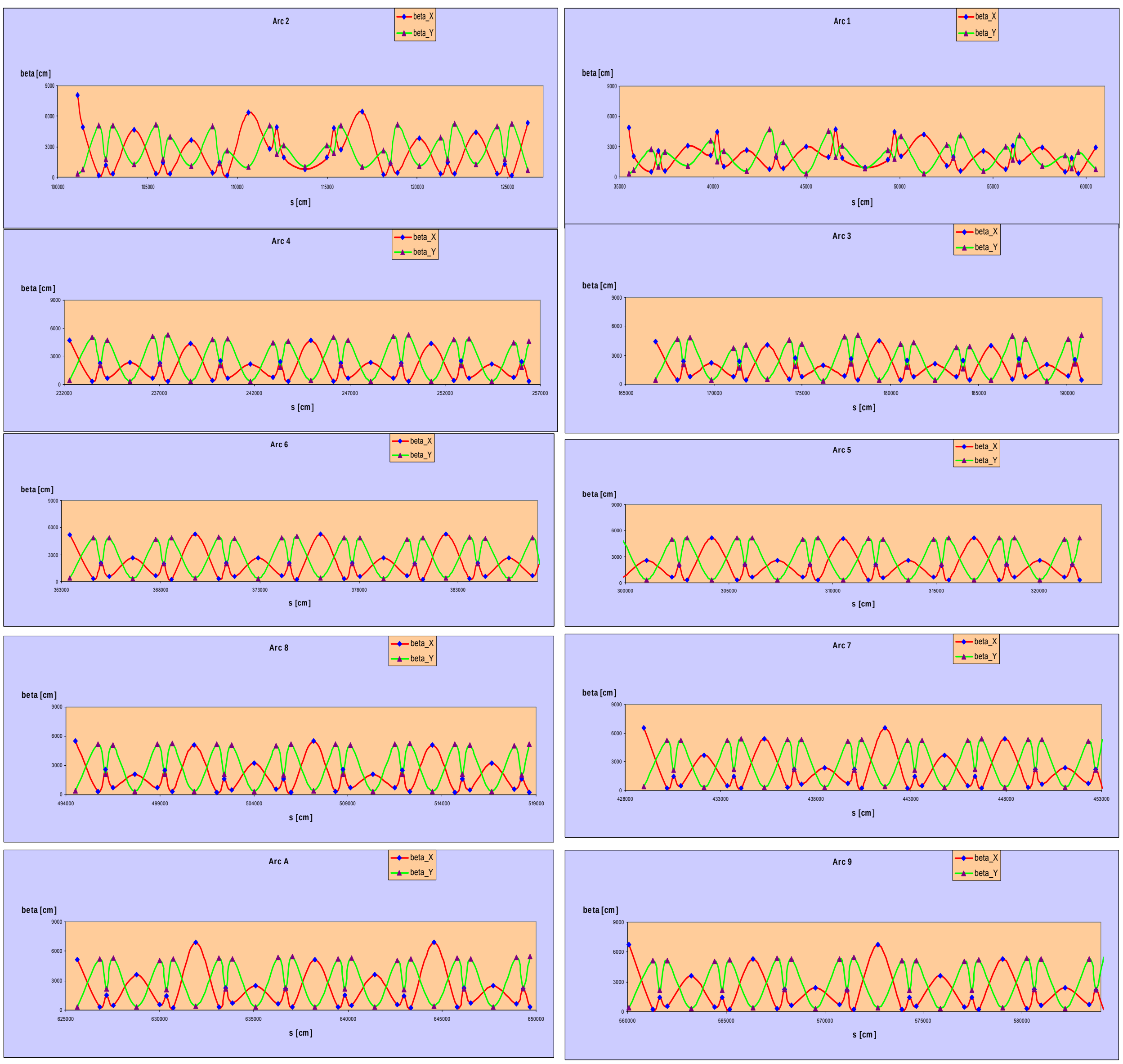

Figure 3 Beta functions for Arcs 1-10

section	quad type	$\sqrt{\sum_{n=1}^{N}\beta_n^2}$ $[cm]$	$\Delta\phi_1\left[cm^{-1}\right]$	$\phi_1^{max}\left[cm^{-1}\right]$	$\Delta\phi_1/\phi_1^{max}$	ε_x $[cm\ rad]$	$\langle\beta_x\rangle$ $[cm]$	σ_x $[cm]$	$\Delta\phi_2$ $[cm^{-2}]$	$\Delta\phi_3$ $[cm^{-3}]$	$\Delta\phi_5$ $[cm^{-5}]$	$\Delta\phi_9$ $[cm^{-9}]$
NL(2-pass)	MQB	50244	1.41E-06	6.48E-04	2.17E-03	2.0E-08	10855	1.5E-02	1.2E-05	1.7E-05	5.8E-05	1.4E-03
SL(1-pass)	MQB	10551	6.70E-06	1.85E-03	3.62E-03	3.0E-08	2179	8.1E-03	1.1E-04	1.7E-04	6.8E-04	2.3E-02
Arc1	MQB	10298	6.87E-06	3.18E-03	2.16E-03	4.0E-08	2138	9.2E-03	9.5E-05	1.5E-04	5.8E-04	1.8E-02
Arc2	MQC	13515	5.23E-06	3.65E-03	1.43E-03	3.0E-08	2339	8.4E-03	8.1E-05	1.3E-04	5.0E-04	1.6E-02
Arc3	MQA	13320	5.31E-06	3.05E-03	1.74E-03	2.0E-08	2822	7.5E-03	9.3E-05	1.5E-04	6.0E-04	2.0E-02
Arc4	MQA	14392	4.91E-06	2.97E-03	1.65E-03	2.0E-08	3027	7.8E-03	8.3E-05	1.3E-04	5.3E-04	1.8E-02
Arc5	MQA	15036	4.70E-06	3.17E-03	1.49E-03	3.0E-08	3169	9.8E-03	6.2E-05	9.4E-05	3.7E-04	1.1E-02
Arc6	MQA	14367	4.92E-06	3.05E-03	1.61E-03	7.0E-08	3022	1.5E-02	4.1E-05	5.9E-05	2.1E-04	5.0E-03
Arc7	MQA	15568	4.54E-06	3.25E-03	1.40E-03	1.2E-07	3272	2.0E-02	2.6E-05	3.6E-05	1.1E-04	2.2E-03
Arc8	MQA	15178	4.66E-06	3.06E-03	1.52E-03	1.9E-07	3186	2.5E-02	2.1E-05	2.7E-05	7.5E-05	1.2E-03
Arc9	MQA	15552	4.55E-06	3.25E-03	1.40E-03	3.6E-07	3269	3.4E-02	1.3E-05	1.6E-05	3.6E-05	4.0E-04
ArcA	MQA	15552	4.55E-06	3.25E-03	1.40E-03	5.0E-07	3269	4.0E-02	1.1E-05	1.2E-05	2.5E-05	2.2E-04

Table 3 Quadrupole magnet specs (integrated gradient, sextupole, octupole, duodecapole and isacopole errors)

section	dipole type	$\sqrt{\sum_{n=1}^{N}\beta_n^2}$ $[cm]$	$\delta\phi_1\left[cm^{-1}\right]$	$\delta\phi_1/\phi_1^{max}$	ε_x $[cm\ rad]$	$\langle\beta_x\rangle$ $[cm]$	σ_x $[cm]$	$\Delta\phi_2$ $[cm^{-2}]$	$\Delta\phi_4$ $[cm^{-4}]$
Arc1	MBE	5664	1.25E-05	3.93E-03	4.00E-08	1935	0.009	1.83E-04	5.34E-04
Arc2	MBR	7052	1.00E-05	2.74E-03	3.00E-08	2412	0.009	1.53E-04	4.48E-04
Arc3	MBE	8072	8.76E-06	2.87E-03	2.00E-08	1896	0.006	1.90E-04	5.89E-04
Arc4	MBB	8292	8.53E-06	2.87E-03	2.00E-08	1881	0.006	1.86E-04	5.77E-04
Arc5	MBB	8667	8.16E-06	2.58E-03	3.00E-08	1951	0.008	1.40E-04	4.18E-04
Arc6	MBB	8951	7.90E-06	2.59E-03	7.00E-08	1945	0.012	8.46E-05	2.30E-04
Arc7	MBA	9915	7.13E-06	2.19E-03	1.20E-07	2093	0.016	5.36E-05	1.33E-04
Arc8	MBA	8890	7.95E-06	2.60E-03	1.90E-07	1949	0.019	4.75E-05	1.09E-04
Arc9	MBA	9879	7.16E-06	2.20E-03	3.60E-07	2088	0.027	2.76E-05	5.38E-05
ArcA	MBA	9879	7.16E-06	2.20E-03	5.00E-07	2089	0.032	2.23E-05	3.97E-05

Table 4 Dipole magnet specs (integrated quadruople body gradient, sextupole and decapole)