Large Distance Effects of Graviton-Graviton Interaction, and Implications for Dark Matter and Dark Energy.

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Abstract: This note details the work published in Ref. [1] where we discuss the possible consequences of non-linear effects in gravity for large mass systems. To qualitatively explore these possible consequences, we draw a parallel between, in the one hand, well known non-linear effects in QCD and, in the other hand, gravity. We then carry a numerical calculation using a simplified Lagrangian in order to quantitatively estimate the effects for gravity. Bearing in mind the approximations of the model and of the calculation technique, the cosmological observations that lead to the hypothesis of existence of Dark Matter can be reproduced without the need for Dark Matter: No exotic ingredients, such as Dark Matter or modifications of gravitational laws, are needed to reproduce the rotation curves of spiral and dwarf spiral galaxies, and to account for the unseen extra mass in clusters of galaxies. Furthermore, our model does not use any arbitrary values for its parameters or unjustified extra-parameters: the form of the Lagrangian is the simplest possible and can be derived from the Einstein-Hilbert Lagrangian of General Relativity, and the values for the coupling constants used are close to the ones expected from the studied system. Non-linear effects also offer a possible natural explanation for the recent observations on the accelerated expansion of the universe, that would otherwise demand the existence of Dark Energy. In that case, our model does not, however, permit us a quantitative estimate.

The parts of this note in green give details of the simulation. These parts can be skipped during a first reading.

1 Introduction

Explanations of important cosmological observations, such as inference of Dark Matter or else the acceleration of the universe, seems to require ingredients exotic to the standard model of fundamental physics. For example, yet undiscovered particles for Dark Matter, or Dark energy for the universe acceleration. However, similar phenomenons are also observed
at smaller distance scales corresponding to the nucleon size. These phenomena, closely related to quark confinement, are fully explained by the standard gauge theory of strong nuclear force (Quantum Chromodynamics, QCD), in particular by its non-abelian nature. In this note, we detail the work done to investigate whether the above cosmological observations could result from similar causes. This work was published in Ref. [1].

A basic feature of the strong nuclear force is confinement of quarks in hadrons. There are two key ingredients to quark confinement. The first is that the QCD coupling $\alpha_s$ is large at the scale of the hadron size. The second ingredient is that QCD is a non-abelian theory (or non-linear theory, in the sense that the field superposition principle does not apply). QCD is non-abelian because the carriers of the force - the gluons - are directly interacting with each other, in contrast e.g. to the carriers of the electromagnetic force - the photons - that are chargeless. Consequently, at a distance of about a Fermi, the strong force is constant with the distance $r$ rather than displaying an $\alpha_s(r)/r^2$ behavior, see e.g. [2].

Similarly, gravity is non-abelian: although massless, gravitons interact with each others because of the mass-energy equivalence. However, the gravity coupling constant $G$ is very small, so non-linear effects are usually negligible: the gravity pull of Earth is the linear sum of the pulls from the Earth components.

Gluons have an odd spin (spin 1) while gravitons have an even spin (spin 2). Consequently, while the strong force can attract or repulse, gravity always attracts and, for a system of mass $M$, $GM$ is more relevant as the coupling constant than $G$. Since $GM$ can be large, non-linear effects may not be negligible anymore. Possible consequences of non-linear effects in gravity were discussed on a heuristic basis in [5] following the similarities between gravity and QCD. If these effects increase significantly the effective strength of the force at large distances, as is the case for QCD, such non-linear effects can be mimicked by either extra mass (Dark Matter in galaxies [6]) or by a modified gravity law [7]. However, if so, Dark Matter or gravity modifications would be ad-hoc fixes that do not encompass the real phenomenon. We note that such non-linear effects would deepen the potential well associated with galaxies and thus be observable from gravitational lensing. A non-abelian nature of gravity is compatible with the equations of general relativity since they are non linear, as already mentioned in [5]. In fact, general relativity can be viewed as a special non-abelian theory [4].

Because non-linear effects happen where asymmetric distributions of baryonic matter are localized (e.g. spiral galaxies, group of galaxies) but should not appear where baryonic matter is uniformly distributed (e.g. gas in galaxy cluster), our model is naturally compatible with the observation of the Bullet cluster observation [20]. Such observation is viewed as a direct proof of Dark Matter modification since it is difficult to interpret it in term of modified

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1: Maybe more accurately: If a gauge theory of gravity exists, it would be non-abelian. The gauge symmetry for general relativity is the general coordinate transformation, as can be seen since gravity couples to impulsion-energy, the conserved quantities stemming from space-time translation invariance. The general coordinate transformation group is non-abelian, see e.g. [3]. In another words, the conserved quantity being the energy-momentum tensor, two generators of the symmetry group would not commute.
gravity, while natural to interpret within the Dark Matter hypothesis. The fact that our model is also naturally compatible with the Bullet cluster observation, while MOND is not, underlines the importance of being conceptually right about the origin of modification of the (effective) strength of gravity.

In this note, a model is proposed to quantitatively study non-abelian effects in gravity. We start from a simple Lagrangian $\mathcal{L}$ that is motivated by:

1. The form of General-Relativity Lagrangian under a weak field approximation.
2. The simplest possible form of Lagrangian that can include the non-linear effects of General Relativity.

We then compute numerically on a grid the potential between two points, with an effective coupling constant $\propto GM$, where $M$ is the typical mass of a galaxy. We then extend our results to a homogeneous disk that at first order could represent a spiral galaxy, and compute the rotation curve of such a "galaxy". We then discuss the implications of our result for Dark Matter and Dark Energy.

2 Model

2.1 Choice of Lagrangian

2.1.1 Derivation from the Einstein-Hilbert Lagrangian

(We use units such as $\hbar = 1$ and $c = 1$.) The Einstein-Hilbert Lagrangian, from which General Relativity can be derived is:

$$\mathcal{L}_{\text{EH}} = \frac{1}{16\pi G} \sqrt{-g} g^{\mu\nu} R_{\mu\nu}$$

where $g_{\mu\nu}$ is the metric, $g = \det g_{\mu\nu}$ and $R^{\mu\nu}$ is the Ricci tensor. The Einstein-Hilbert action is $$\int d^4x \mathcal{L}_{\text{EH}}.$$ For weak fields, the Einstein-Hilbert action can be rigorously expanded in a power series of the coupling $k (k^2 \propto G)$ by developing the metric $g_{\mu\nu}$ around the flat metric $\eta_{\mu\nu}$. This is known (see e.g. refs. [8], [9], [10]) but we recall it for convenience: $g_{\mu\nu}$ is parametrized, e.g. $g_{\mu\nu} = (e^{k\psi})_{\mu\nu}$, and expanded around $\eta_{\mu\nu}$. It leads to:

$$\frac{1}{16\pi G} \int d^4 x \sqrt{-g} g_{\mu\nu} R^{\mu\nu} = \int d^4 x (\partial\psi\partial\psi + k\psi^2\partial\psi + k^2 \psi^2\partial^2\psi + ... ) + k\psi_{\mu\nu}T^{\mu\nu} \quad (1)$$

Here, $\psi^{\mu\nu}$ is the gravity field, and $T^{\mu\nu}$ is the source (stress-energy) tensor. Since our interest is $\psi$ self-interactions, we will not include the source term in the action. (We note that it does
not mean that $T^{\mu\nu}$ is negligible: we will use later the fact that $T^{00}$ is large. It means that $T^{\mu\nu}$ is not a relevant degrees of freedom in our specific case. This will be further justified later.) A shorthand notation is used for the terms $\psi^n \partial \psi \partial \psi$ which are linear combinations of terms having this form for which the Lorentz indexes are placed differently. For example, the explicit form of the shorthand $\partial \psi \partial \psi$ is given by the Fierz-Pauli Lagrangian [11] for linearized gravity field: $\partial \psi \partial \psi = \frac{1}{2} \partial \lambda \psi^{\mu\nu} \partial \lambda \psi_{\mu\nu} - \frac{1}{2} \partial \lambda \psi^\mu_\mu \partial \lambda \psi^\nu_\nu - \partial \lambda \psi_{\lambda\nu} \partial \psi^\nu_\mu - \partial \lambda \psi^\mu_\lambda \partial \psi^\nu_\mu$.

The Lagrangian $\mathcal{L}$ is a sum of $\psi^n \partial \psi \partial \psi$. These terms can be transformed into $\frac{1}{n+1} \psi^{n+1} \partial^2 \psi$ by integrating by part in the action $\int d^4x \mathcal{L}$. We consider first the $\partial \psi \partial \psi$ term. The Euler-Lagrange equation of motion obtained by varying the Fierz-Pauli Lagrangian leads to $\partial^2 \psi^{\mu\nu} = -k^2 (T^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} Tr(T))$. We ignore the $-\frac{1}{2} \eta^{\mu\nu} Tr(T)$ for now and have $\partial^2 \psi^{\mu\nu} \propto T^{\mu\nu}$. Since the $T^{00}$ component dominates $T^{\mu\nu}$ within the stationary weak field approximation, so too $\partial^2 \psi^{00}$ dominates $\partial^2 \psi^{\mu\nu}$ and one can keep only the $\psi^{00}$ terms in $\psi \partial^2 \psi$, i.e. in $\partial \psi \partial \psi$. Finally, after applying the harmonic gauge condition $\partial^\mu \psi_{\mu \nu} = \frac{1}{2} \partial \nu \psi^\nu_\xi$, we obtain for the first term in $\mathcal{L} \partial \psi \partial \psi - \frac{1}{2} \partial \lambda \psi^{00} \partial \lambda \psi_{00}$. The $-\frac{1}{2} \eta^{\mu\nu} Tr(T)$ term we ignored does not change this conclusion: $\partial^\mu \psi_{\mu \nu} = -k^2 (T^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} Tr(T))$ leads (using $\eta_{00} = +1$) to $\partial^2 \psi_{00} = -8\pi G T_{00} = -\partial^2 \psi_{ii}$, with $i = 1$ to 3. Developing the sums in the first term of the Fierz-Pauli Lagrangian, we get $\partial \lambda \psi^{\mu\nu} \partial \lambda \psi_{\mu \nu} = \partial \lambda \psi^{00} \partial \lambda \psi_{00} + \partial \lambda \psi^{11} \partial \lambda \psi_{11} + ...$ and, after integrating by part in the action: $\partial \lambda \psi^{\mu\nu} \partial \lambda \psi_{\mu \nu} = \psi^{00} \partial^2 \psi_{00} + \psi^{11} \partial^2 \psi_{11} + ... = -2 \psi^{00} \partial^2 \psi_{00}$ since all the $\partial^2 \psi_{\mu \nu}$ terms are proportional to each other. Similarly the second term in the Fierz-Pauli Lagrangian becomes $\partial \lambda \psi^{\mu\nu} \partial \lambda \psi_{\nu \nu} = \partial \lambda \psi^{00} \partial \lambda \psi_{00} + \partial \lambda \psi^{11} \partial \lambda \psi_{11} + ... \psi^{00} \partial^2 \psi_{00}$, and so on for the other terms of the Fierz-Pauli Lagrangian.

Higher order terms in Eq. 1 should proceed similarly since they are all of the form $\frac{1}{n+1} \psi^{n+1} \partial^2 \psi$. The factor in front of each $\psi^n \partial \psi \partial \psi$ ($n \neq 0$) may depend however on how $g_{\mu \nu}$ is expanded around $\eta_{\mu \nu}$. For this reason, and because the higher order terms are complicated to derive, we use a different approach to determine the rest of the Lagrangian.

### 2.1.2 Derivation à la Landau-Ginzburg

We build the Lagrangian from the appropriate Feynman graphs (see Fig. 1) using, with hindsight of previous discussion, only the $\psi^{00} \equiv \phi$ component of the field. Each term in the Lagrangian corresponds to a Feynman graph: the terms quadratic, cubic and quartic in $\phi$...
\( \phi \) correspond respectively to the free propagator, the three legs contact interaction and the four legs contact interactions. The forms \( \phi \partial_\mu \phi \partial_\nu \phi \) and \( \phi^2 \partial_\mu \phi \partial_\nu \phi \) (rather than \( \phi^3 \) or \( \phi^2 \partial_\mu \phi \) for example for the three legs graph) are imposed by the dimension of \( G \). (Note that in Eq. 1, the origin of the two derivatives in the generic form \( \phi^n \partial_\mu \phi \) is from the two derivatives in the Ricci tensor and the absence of derivative in \( g_{\mu\nu} \)). We chose the Lagrangian \( L \) leading to the following action:

\[
S = \int d^4x L = \int d^4x \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 + \frac{g'}{3!} \phi \partial_\mu \phi \partial_\nu \phi + \frac{g}{4!} \phi^2 \partial_\mu \phi \partial_\nu \phi.
\]

(2)

This is the simplest choice for \( L \) that accommodates the dimensions of the effective coupling constants \( g \propto g'^2 \propto G \). The \( g \neq 0 \) and \( g' \neq 0 \) gives rise to the non-abelian effects. The factors 3! and 4! cancel the combinatorial factors that arise during the calculation of the green function when a Wick contraction is done. Hence, each Feynman graph associated with a \( \phi^n \partial_\mu \phi \partial_\nu \phi \) term has a coupling \( G^{(n-2)/2} \) at an \( n \)-legs vertex rather than \( n!G^{(n-2)/2} \).

The mass \( m \) would be the graviton mass if \( L \) was the actual Lagrangian for gravity. This mass should be taken to 0 but cases for which \( m \neq 0 \) are useful for testing our model. The \( \partial_\mu \) in the \( g' \) and \( g \) terms provide the correct units for \( g \) and \( g' \). The \( g' \) term is breaking the \( \phi \) symmetry of \( L \). It is, however, needed to allow for the contribution of odd-legged graphs, present in QCD and in gravity, of the type shown in Fig. 2. The 3! and 4! are the usual combinatorial factors introduced in connection with the Feynman graphs corresponding to the Lagrangian. A similar Form of Lagrangian but using the proper tensor fields can be found e.g. in [10], 8 or [9].

Since we are interested in the instantaneous (Coulomb-like) potential of a system in dynamical equilibrium, space-time is reduced to 3 space dimensions and a \( S \times S \times S \) cubic grid is used. With this Lagrangian, the dimensions of the effective coupling constants are \([g] = \text{GeV}^{-1}\) and \([g'] = \text{GeV}^{-1/2}\). A Monte-Carlo Metropolis algorithm is employed to estimate the 2-point correlation function (Green function) using its path integral expression \( V(r_1 - r_2) = \int Dx \phi(r_1)\phi(r_2)e^{-S} \). The correlation function gives the potential between the two points \( r_1 \) and \( r_2 \).

At the origin of the model, it is assumed that using one field and the large effective coupling constants \( g \propto GM \) and \( g' \propto \sqrt{GM} \) is equivalent to having \( M/m \) fields (here \( m \) is the typical mass of the elementary components, e.g. a typical star mass, or else the nucleon...
mass\(^4\) and the gravity coupling constants \(g \propto G\) and \(g' \propto \sqrt{G}\). To be consistent with this hypothesis, we set \(g' = \sqrt{g}\).

We use \(GM\) as an effective coupling constant rather than \(GMm\), where \(M\) is the total mass of the system and \(m\) is a typical star mass, because the effective coupling constant is equivalent to a sum of fields. For example, if we have an object made of \(m\) elementary components interacting with an object made of \(n\) elementary components and each component emits \(\alpha\) gravitons, we have \(\alpha(m+n)\) gravitons, not \(\alpha(mn)\). The additive form of the global coupling is supported by the \(G^2\) correction to the precession of the perihelion of the orbit of two bodies of masses \(m_1\) and \(m_2\), that is given by \(\frac{Gm_1m_2}{r} \left( \frac{G(m_1+m_2)}{2r} \right)\), or in dominant one-loop correction to the Newtonian potential established in quantum gravity (i.e. the first order graviton self-interaction):

\[
V(r) = \frac{Gm_1m_2}{r} \left( 1 + \frac{3G(m_1+m_2)}{2r} \right) [9].
\]

Comparison of these corrections to our global coupling are relevant because they stem in part from the field-self interaction. Clearly, our estimate of the magnitude of the global coupling is naive. In particular, the use of the field superposition principle is inadequate when large non-linear effects are present.

### 2.2 Calculation of a Yukawa-type case

As a first check, we verify that a potential \(V(r) \propto \frac{(e^{-mr})}{r}\) is obtained when \(g = g' = 0\). Results are shown on Fig. 3 for various values of \(m\). There is an excellent agreement between the simulation and the expected potential.

Other results are shown on Fig. 4 for various values of parameters of the simulation, such as:

- \(S\), the size of the grid.
- \(N_{cf}\), the number of configurations used to calculate the correlation function.
- \(N_{cor}\), the number of unused configuration updates between each calculation of the correlation function (\(N_{cor}\) insures that the \(N_{cf}\) configurations are decorrelated).

In the top plot, the effect of statistics (\(N_{cf}\)) and possible correlations between configurations (\(N_{cor}\)) are checked for two different grid sizes (the value of the grid spacing \(d\) stays the same). Some small disagreements appear, possibly due to too low values of \(N_{cor}\) or \(N_{cf}\). In the 2\(^{nd}\) plot, the effect of the grid size is checked (for constant \(d\)). This is checked again on the 3\(^{rd}\) plot for higher statistics and decorrelation parameter \(N_{cor}\). The bottom plot shows the agreement for various higher values of \(N_{cf}\) and \(N_{cor}\). From this plot, it appears that \(N_{cor} = 10\) and \(N_{cf} = 10^4\) can be used for the simulation.

\(^4\)It is equivalent to take the nucleon or the star as elementary components because gravitational non-
Figure 3: Simulated results in the abelian case for different masses $m$, compared to the expected $V(r) \propto (e^{-mr})/r$. Residuals are shown on the right.
Figure 4: Comparisons of results for various values of parameters. For these simulations, the spacing $d$ is kept constant while the system size is varied.
2.3 Check that result is independent of grid spacing \( d \)

In the section 2.2, we checked in particular that for constant grid spacing \( d \) but for different grid sizes, we obtained consistent results. We check here that we also obtain consistent results when \( d \) is varied and the grid size remains constant.

2.3.1 Coulomb-like potential

We first check the Abelian case with \( m = 0 \), using the results already shown in Fig. 4. We scale \( r \rightarrow r \times d \). The green function is quadratic in the field (it is of the form \( V \sim \int D\phi_1 \phi_2 \)). For \( \mathcal{L} \) in 3 dimensions, the dimension of the field is \([\phi]=1дается.distance^{-1/2}\) so \( V \) is scaled as \( d^{-1} \).

The results are shown on Fig. 5 and display good agreement. In the top plot, we scale the results for size \( = 60 \) (that is a \( 60 \times 60 \times 60 \) cube) and 72 so that they correspond to those for size \( = 36 \). In the bottom plot, we scaled results for size \( = 60 \) and 36 so that they correspond to those for size \( = 24 \).

This simple scaling could have been guessed immediately without looking at \( \mathcal{L} \) and the form of the Green function since a Coulomb-like potential varies as \( V \propto 1/r \). This scaling works only because \( m = 0 \). In the next section, we will check the case with \( m \neq 0 \).

2.3.2 Yukawa potential

In the abelian case with \( m \neq 0 \), we cannot simply scale \( V(r) \) but have to rescale the dimensionful terms in the 3D version of Eq. 2. To have the action dimensionless, we scale as:

- \( \phi \rightarrow \phi \times \sqrt{d} \)
- \( m \rightarrow m \times d \)
- \( V \rightarrow V/d \)

Results for various values of \( d \) are in good agreement and are shown in Fig. 6.

\[ \text{abelian effects are negligible in both cases. Consequently, the star mass is approximately the linear sum of the masses of the nucleons constituting the star (with additional electron masses and small corrections for nuclear binding energy). We could not take quarks as elementary components since their non-abelian effects are large and the binding energy contributes largely to the mass of the quark system (i.e. the nucleon) while the quark mass \( m_q \) contribute very little to it: } \sum m_q \approx 15 \text{ MeV} \ll m_N \approx 940 \text{ MeV}. \]

\( ^5 \)This implies \( g' \propto \sqrt{MG} \). The choice \( g' \propto M\sqrt{G} \) would also seems to be a possible hypothesis but in that case, \( g' \) would be dimensionless. In that case, non-abelian effects would be large for \( M \sim 1/\sqrt{G} = 1.2 \times 10^{19} \text{ GeV} \equiv 2 \times 10^{-8} \text{ kg} \), which is clearly not true.
Figure 5: Verification in the abelian and $m = 0$ case that the numerical calculation results for different grid spacings $d$ are consistent. In the top plot, we scaled the results of Fig. 4 to a system size of 36 with grid sizes of 36 ($d = 1$, green triangles), 60 ($d = 36/60$, blue circles) and 72 ($d = 36/72$, pink squares). In the bottom plot, we scaled to a system size of 24 with grid sizes of 24 ($d = 1$, blue down triangles), 36 ($d = 24/36$, up red triangles) and 60 ($d = 24/60$, black squares). All results are consistent.
Figure 6: Results for the Coulomb-like case $m = 0$, $d = 1$ (blue symbols) and $m = 0$, $d = 0.5$ (green symbols). Results for the Yukawa case for $m = 0.4$ are shown in black ($d = 1$) and red ($d = 0.5$).
2.3.3 Non-abelian case

We checked that the results are independent of the grid size for two different sets of values for the effective coupling constants \( g \) and \( g' \). Results are shown in Fig. 8. In the left plot, we used \( g' = \sqrt{g} = 4.2 \times 10^{-3} \) and grid sizes from 36 to 98 (with \( N_{\text{cor}}=7 \) and \( N_{\text{cf}}=5000 \)). On the right plot, we used \( g' = \sqrt{g} = 8.4 \times 10^{-3} \) and grid sizes from 24 to 72 (with \( N_{\text{cor}}=5 \) and \( N_{\text{cf}}=5000 \)). To obtain these results, we first run the program without rescaling the dimensionful quantities in \( L \). Then, to scale the potential, we first subtracted the abelian (coulomb-like) contribution from the results. We then scaled the remaining by \( d^4 \). We removed the abelian contribution because it scales differently than the non-abelian one (scales as \( d^{-1} \) as discussed in section 2.3.1). The reason for scaling by \( d^4 \) is because we have to scale a linear potential. It can be explained as following: A factor \( d \) comes from the Green function being bilinear in the field, with the field scaling as \( \sqrt{d} \). Another factor \( d \) comes from scaling the distances in the direction along the two points between which the potential is calculated (for e.g., a string produces a potential \( \alpha r \). This potential should be scaled down by \( d \) when \( r \) is scaled up by \( d \). A factor \( d^2 \) comes from scaling the distances in the two perpendicular directions (the potential is proportional to the amount of flux lines going through a unit area. If we scale the perpendicular distances, we scale the amount of flux lines by a factor \( d^2 \), see Fig. 7). The results are in good agreement for \( \text{size} > 36 \). For sizes 36 or less, it appears that finite size effects are still important.

An equivalent explanation is that for a large effective coupling constant all field lines that would, in the abelian case, fill a volume \( \frac{4}{3} \pi d^3 \) are being concentrated between the two points. Consequently, the strength should scale by \( (d)^3 \).

The \( d^4 \) scaling law is true only when \( g' \) is large enough so that the resulting potential is linear. For \( g' \) smaller, the scaling law would be an intermediate case between the abelian and large \( g' \) cases.

The fact that the two components of the potential scales differently poses a problem: Our procedure to obtain an absolute result is to normalize the non-abelian case to the (known) abelian case. This is done for ex. in Eq. 4 (to compare to Eq. 3). However, because the two potentials scale differently, an arbitrary dependence to the system size still remains. A distance scale is needed to determine the constant \( \kappa \) that would fix this arbitrary dependence. This scale can be chosen, e.g., as the typical scale for which non-abelian effects become important. For spiral galaxies, this is typically a kpc. With a grid of \( n \times n \times n = 48^3 \) nodes and with the distance unit \( d \) between each nodes being 1 kpc, the distance scale at which non-abelian effects become important can be reproduced. Consequently, \( \kappa \sim 1 \) in this cases.

For results from a \( n \times n \times n = 60^3 \) grid, \( \kappa \sim 0.6 \). The \( g \to g' \left(\frac{\alpha}{\frac{4}{3} \pi d^3}\right)^2 \) scaling works for a given physical system size \( L \). If both the grid unit and the system size are varied, ex as \( d \to d/\lambda \) and \( L \to \lambda L \), then another scaling factor \( L^3 \) is necessary to insure that our results are independent of our choice of \( d \). We assume that the factor is related to the volume \( \frac{4}{3} \pi L^3 \).

We tentatively explain it in the following way: for fixed \( L \), varying \( d \) increases the number of nodes, i.e. of fundamental oscillator elements, in the grid since \( (L/d)^3 \) gives the number of
nodes. This produces the \((\frac{4}{3}\pi d^3)^2\) rescaling factor. When the physical system size \(L\) is itself varied while the number of nodes stay constant, the amount of particles carrying the force is diluted as \(L^3\), and this may explain the \(L^3\) scaling factor.

All in all, \(g = 16\pi GM \rightarrow g \times \frac{1}{4}(\frac{4}{3}\pi d^3)^2L^3\) where the \(1/d\) converts \(g\) into the grid unit and \((\frac{4}{3}\pi d^3)^2\) accounts for the grid volume scaling discussed above. With size = \(\frac{L}{d}\) being the size of the grid, \(g\) scales overall as \(g \rightarrow g(\frac{4}{3}\pi)^3\text{size}^{-5}L^2\). The constant factor \((\frac{4}{3}\pi)^3\) will be now included in the definition of \(g\).

The cancellation of large quantities \((M, d, G)\) yields a coupling constant of moderate value. This could induce a large uncertainty in the model results. Furthermore there is assumptions on the form and the scaling of the effective coupling constants. A more conservative, but less appealing, approach would be to consider \(g' = \sqrt{g}\) as a free parameter of the model.

### 2.4 Simulation results

Results for non-zero \(g\) and \(g'\) are shown in the left side of Fig. 9. These results are for \(48 \times 48 \times 48\) grid systems of size \(L = 28d\), which, for \(d = 1\) kpc \((= 1.57 \times 10^{35}\) GeV\(^{-1}\)), correspond to a typical galaxy diameter of about 45 kpc (It fits in a \(28 \times 28 \times 28\) cube of grid spacing \(d\): \(45 \sim \sqrt{3(28-2)^2}\). The “\(-2\)” is because nodes on the grid boundary are not counted since they are not updated, see section 2.5.). We use \(g' = \sqrt{\frac{16\pi GM}{d}} = 4.9 \times 10^{-3}\) as a typical value for the effective coupling constants. The factor \(d\) converts the coupling constant to grid unit. We used \(M = 10 \times 10^9\) M\(_\odot\) = \(1.12 \times 10^{67}\) GeV \([13]\) as a typical galaxy mass. For \(r \gtrsim 5\), the non-abelian result displays a roughly linear behavior. The other result is for \(g = 0\) and displays the expected \(1/r\) form. In the top right plot of Fig. 9, the result for \(g = 0\) is subtracted from the result for \(g \neq 0\). We fitted this result (thick black line). The derivative of this fit, together with the derivative of the fit to the \(g = 0\) results, are shown on the right bottom plot by the continuous and dashed lines, respectively. The derivative representing the force between the two points, our model indicates that non-linear effects start to dominate for a distance 1.9 kpc for the chosen values of \(g\) and \(g'\) and for a 2-point system.
Figure 8: Scaling in the non-abelian case for two different values of the coupling constant (Left: $g' = 4.2 \times 10^{-3}$, right: $g' = 8.4 \times 10^{-3}$). Finite size effects seems to be present for size $\leq 36$.

### 2.5 Boundary conditions

Usual circular boundary conditions were not used in the simulation for the following reason: For a Lagrangian that would lead to a strictly linear potential $V = -ax$, the contribution from the next half of the grid pattern for $x' = S + x$ would be $V = a(-x + S)$. The total simulation result would return an irrelevant constant $aS$ rather that the linear form $V = -ax$. Instead of circular boundary conditions, we set the boundary nodes of the grid to be random with an average 0 value. These nodes are never updated. In addition, although we update the fields on the nodes close to the boundary nodes, we do not use them in the calculation of the correlation function (in the results presented, we ignored the 4 nodes closest to the grid boundary. We varied this number and found compatible results). We obtain excellent agreement for various grid sizes, see Figs. 4 and 9. Caution is, however, necessary for the $g' \neq 0$ case when $g'$ is large enough: with a linear potential, the correlations between distant points never disappear, and boundary conditions always influence the results, no matter how large $S$ is. It is possible that the linear behavior of $V(r)$ is not a true consequence of non-abelian effect but an artifact due to our choice of boundary conditions. They could force the potential to 0, the average value of the fields on the boundary nodes. However, some facts indicate that it may not be the case and that our choice of boundary conditions does not introduce non-physical artifacts:
Figure 9: Left plot: results in the $g = g' = 0$ case (black diamonds) and in the $g' = \sqrt{g} \neq 0$ case (red stars). $r$ is in grid unit. On the top right plot, the result for $g = g' = 0$ is subtracted from the results for $g' = \sqrt{g} \neq 0$. The thick black line is a fit of the subtraction result. Derivatives of this fit (continuous line) and of the fit to the $g = g' = 0$ result (dashed line) are shown on the right bottom plot.
1. The results are self-consistent: to encounter the problem of non-vanishing correlations, we need to have a linear (or any strongly correlated) potential in the first place. So the linearity is there in the first place and not resulting from of particular choice of boundary conditions.

2. We ran the simulation for various grid sizes and/or spacing $d$ and obtained compatible results.

3. We ran the simulation with circular boundary conditions and did find that a non-zero $g'$ or $g$ adds a constant to a potential close to the $e^{-mr}/r$, consistently with the case of a linear potential computed with circular boundary conditions.

4. We ran the simulation with various masses $m$ and extrapolated the results to $m = 0$. In these cases, the influence of the boundary conditions is reduced by a factor $e^{-mr}$ and becomes quickly negligible as in the $g = g' = 0$ case. Although there are uncertainties associated with the extrapolation to $m = 0$, the results are consistent with the linear potential initially found.

The choice of boundary conditions assumes that the system size is the same as the grid size and that the outside field, not contained in the grid, is negligible compared to the field in the system.

3 Extension to homogeneous circle and sphere.

As mentioned in [5], non-abelian effects depend on the geometry of the system. For masses distributed homogeneously with a spherical symmetry, effects are expected to cancel out. This can be pictured as follow: for a two-point system, field lines attract each others and collapse in a “flux tube” centered on the line linking the two points. Such flux tubes are a well known properties of QCD in its confinement regime. For spherical symmetry, all field line attractions compensate each others and there is no departure from the case of the abelian, coulomb-like, field lines. Similarly, cylindrical symmetry should reduce non-abelian effects.

To estimate the effects in the case of cylindrical symmetry, or equivalently of a thin disk that we can consider at first order as a spiral galaxy, we approximate our results for $V(r)$ in Fig. 9 as perfectly linear: All the field lines are straight lines linking the two points and we obtain a force of constant value $b$. In the case of the thin disk, we are interested in the potential between the center of the disk and the circumference points. The field lines are evenly distributed on the circumference, and the force is reduced by a factor $2\pi r$ where $r$ is the radius of the disk. This leads to a force $b/(2\pi r)$ and a potential $b/(2\pi)ln(r)$. Adding back the (unaffected) abelian part, we obtain a potential of the type:

$$V(r) = \frac{a}{r} + \frac{b}{2\pi}ln(r)$$

(3)
This is the same form as one of the potential for modified gravity determined empirically from measurements of rotational curves of galaxies [12]. It is also the same form found in [5] based on heuristics arguments.

In the case of a homogeneous spherical distribution, the constant force becomes \( b/4\pi r^2 \), leading to a potential \((a + b/4\pi)/r = c/r\) that displays a coulomb-like form as expected.

4 Spiral galaxies

4.1 Rotation curves of spiral galaxies

The approximately linear potential obtained from our model and generalized to an homogeneous disk is:

\[
V(r) = -GM \left( \frac{1}{r} + \frac{b}{2\pi a} \ln(r) \right) \tag{4}
\]

where the unit of \( b/(2\pi a) \) is an inverse distance in grid unit. We simulated a cubic grid of \( 48^3 \) nodes. The grid unit is \( d = 1 \) kpc=\(3.09 \times 10^{10} \) m. Fitting the results shown in Fig. 9 we obtain \( a = 0.081 \) and \( b = -0.092 \) for \( M = 10 \times 10^9 \) M\(_\odot\) and a galaxy diameter of \( D = 45 \) kpc. Going from grid unit to meters, we have: \( -b/(2\pi a) = 0.092/(2\pi \times 0.081)/(3.09 \times 10^{19}) = 5.84 \times 10^{-21} \) m\(^{-1}\). With these factors, and assuming a simple exponential decrease of the galaxy density with its radius such as \( \rho(r) = \frac{M}{2\pi r_0} e^{-r/r_0} \) with \( r_0 = 2 \) kpc, we obtain the rotation curve shown in Fig. 10. Also shown in the figure is the typical rotation curve for two galaxies of mass and radius similar to the parameters chosen in our model (NGC 6503 and NGC 2403, data are from ref. [13]) The model (continuous line) reproduces well the magnitude and shape of the rotation curve, without the need for Dark Matter or modification of gravity law. The curve that one would obtain without non-abelian effects (Newtonian potential) is shown by the dashed line.

We compared to other rotation curves of galaxies given in Ref. [13]. The luminous mass and size of a galaxy are not well known. We have adjusted \( M \) in order to fit best the rotation curves. We used the value of \( R_{25} \) given in Ref. [13] for the galaxy radii. The parameter \( r_0 \) giving the exponential decrease of the luminous matter is also adjusted and its values lie between 1 kpc and 2.4 kpc. The results are shown in Fig. 11. Table 1 gives the parameter used in the simulation, and the luminosity \( L \) and size \( R_{25} \) of the galaxies as given in Ref. [13].

The inferred mass \( M \) and luminosity \( L \) are of similar values, and it is expected that \( L \) gives a lower bound for \( M \). The curves reproduce well the data given the simplicity of the model used for a galaxy and our simplified Lagrangian.
Figure 10:
Rotation curve obtained with our model (continuous line) and the curve that would be obtained without non-abelian effects (dashed line). Also shown are rotation curves of galaxies of mass and size similar to the inputs used in the model.

Table 1:

<table>
<thead>
<tr>
<th>galaxy</th>
<th>$M$ ($10^9 M_\odot$)</th>
<th>$r_0$ (kpc)</th>
<th>$L$ ($10^9 M_\odot$)</th>
<th>$R_{25}$ (kpc)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NGC 6503</td>
<td>12</td>
<td>1.8</td>
<td>4.80</td>
<td>22.22</td>
</tr>
<tr>
<td>NGC 2403</td>
<td>14.6</td>
<td>2.6</td>
<td>7.90</td>
<td>19.49</td>
</tr>
<tr>
<td>NGC 2841</td>
<td>36.6</td>
<td>1.2</td>
<td>20.50</td>
<td>42.63</td>
</tr>
<tr>
<td>NGC 2903</td>
<td>21</td>
<td>1.1</td>
<td>15.30</td>
<td>24.18</td>
</tr>
<tr>
<td>NGC 3198</td>
<td>16.2</td>
<td>2.3</td>
<td>9.00</td>
<td>29.92</td>
</tr>
<tr>
<td>NGC 7331</td>
<td>27.8</td>
<td>1.5</td>
<td>54.00</td>
<td>36.72</td>
</tr>
</tbody>
</table>
Figure 11:
Rotation curves obtained with our model (continuous line) compared to the measurements and the curve that would be obtained without non-abelian effects (dashed line). The mass $M$ and the exponential decrease parameter $r_0$ being not well known, they are adjusted to best fit the data.
4.2 The Tully-Fisher Relation

The Tully-Fisher relation [14] is an empirical law that relates linearly the log of a spiral galaxy mass to the log of its rotation velocity \( v \): 

\[
\ln(M) = \alpha \ln(v) - \beta,
\]

with \( \alpha = 3.9 \pm 0.2 \) and \( \beta \approx 1.5 \). The relation is readily explained within our framework: Equating the centripetal force acting on an object of mass \( m \), \(-mv^2\mathbf{u}_r/r\) (\( \mathbf{u}_r \) is the unit vector) to the gravitational force \(-GMmb\mathbf{u}_r/(2\pi a)\) given by the potential of Eq. 4 for large distances yields 

\[
v^2 = \frac{GMb}{2\pi a},
\]

Since \( a \), the coefficient of the Newtonian potential in Eq. 4 is proportional to \( GM \) (\( a = \tau GM \)), and since \( b \) can only be a function of the field self-coupling magnitude, \( b(\sqrt{GM}) \), we have 

\[
v^2 = b(\sqrt{GM})/(2\pi \tau),
\]

which correlates \( v \) and \( M \) since \( G \) is a constant. This qualitatively explains the Tully-Fisher relation. The coefficient \( \alpha \) of the Tully-Fisher relation can be obtained by expanding \( b(\sqrt{GM}) \):

\[
b = b_0 + b_1\sqrt{GM} + b_2GM + \ldots.
\]

Without field self-coupling (i.e. setting \( \sqrt{GM} = 0 \) in Eq. 2), \( b = 0 \) so \( b_0 = 0 \). Since \( \sqrt{GM} \gg GM \) we have at leading order 

\[
\ln(M) = 4\ln(v) + \ln(2\pi \tau/\sqrt{Gb_1}).
\]

We remark that the Tully-Fisher relation is not explained in the dark matter scenario, and is a built-in feature of MOND as are the flat rotation curves. We also remark that a relation akin to the Tully-Fisher one exists for the strong force in the confinement regime: the angular momenta and squared masses of hadrons are linearly correlated. These “Regge trajectories” are at the origin of the string picture of the strong force.

5 Other systems

5.1 Clusters of galaxies

Dark Matter was first hypothesized to reconcile the motions of galaxies inside clusters with the measured luminous mass of the cluster. It is not possible within our model to estimate or infer the net gravitational force between a spiral galaxy and another galaxy because on the one hand the force may be decreased since gravitons may be partially confined in the spiral galaxy. On the other hand, non-abelian effects would increase the effective gravitational force between the two galaxies. However, since galaxy clusters are composed mostly of elliptical galaxies, we assume that the partial graviton confinement can be ignored. In order to carry out calculations within our model, we assume furthermore that the intergalactic gas, which dominates the cluster mass, is distributed homogeneously enough so that it does not influence the computation of non-abelian effects. Finally we also restrict the calculation to the interaction of 2 galaxies, assuming that the others play no role. With these three strong assumptions, the results already obtained in section 2.3.3 for two-body systems can be used. Taking 1000 kpc as the distance between the two galaxies and the total mass of the two galaxies to be \( 40 \times 10^9 \, M_\odot \), we obtain an effective coupling constant \( g = 3.66 \times 10^{-5} \) in grid unit. It produces a roughly linear potential similar to the one of Fig. 9 but with a slope of 

\[
b = -0.012 \text{ in grid unit.}
\]

We can express this non-abelian effect from the point of view of
non-baryonic dark matter by assuming that gravity obeys its Newtonian form. We cast our result into the forms:

\[ V(r) = -\frac{GM}{2} \left( \frac{1}{r} - \frac{b}{a} \right) = -\frac{GM'}{2} \frac{1}{r} \]

where the right-hand side of the equation forces a Newtonian potential form, and the non-abelian effects appear as an extra mass \((M' - M)\). We have the ratio:

\[ \frac{M'}{M} = 1 - \frac{b}{a} r^2 = 251. \]  

Recalling that gaseous mass in a cluster is typically 7 times larger than the galaxy mass, and assuming that half of the galaxies in the cluster are spirals or flat elliptical for which we can neglect the effect while the others are perfectly spherical galaxies for which the calculation above applies, we obtain for the cluster a ratio \(\frac{M'_{\text{cluster}}}{M_{\text{cluster}}} = 18.0\). That is non-abelian effects would make our model of cluster appear to be composed of 94% of non-baryonic Dark Matter, to be compared with the typical 80-95% inferred from observations.

5.2 Spiral dwarf galaxies

The results of our model for two dwarf galaxies (DDO 170 and DDO 153) are shown in Figure 12. The model reproduces well the fact that only a small fraction of the rotation curve can be attributed to a Newtonian potential with the measured luminous mass. From the Dark Matter standpoint, dwarf spiral galaxy masses are essentially coming from Dark Matter halos. The mass and \(r_0\) inputs for DDO 170 are \(0.54 \times 10^9 \, M_\odot\) and 2 kpc (compared to \(L = 0.16 \times 10^9 \, M_\odot\)) with \(R_{25} = 2.41\) kpc [13]. The mass and \(r_0\) inputs for DDO 153 are \(0.29 \times 10^9 \, M_\odot\) and 1.7 kpc (compared to \(L = 0.05 \times 10^9 \, M_\odot\)) with \(R_{25} = 1.05\) kpc [13].

5.3 Sun-Pioneer system

We applied our model to the sun-Pioneer 2-body system to see if it reproduces the Pioneer anomaly [15]. We used a total system mass of \(M = 1 \, M_\odot\) and a grid spacing of \(1.57 \times 10^{11}\) m \((5.09 \times 10^{-9}\) kpc). This results in an effective coupling constant \(g = 4.69 \times 10^{-7}\). Since we are close to the abelian case the \((\frac{4}{3}\pi)^3\) is not included in the definition of \(g\), see section 2.3.3. We
computed the potential at a distance of 40 AU from the sun. However, the systematic and statistic uncertainties associated with the model result do not allow us to isolate any non-zero non-abelian effects. The statistical uncertainty is at the level of $10^{-8}$ while the systematics is at least at the $10^{-6}$ level, to compare with the pioneer anomaly at $10^{-9}$ level.

6 Dark Energy

The second basic feature of the strong force is that once quarks are confined in hadrons, these quark systems (i.e. hadrons) do not interact with each other through the strong force, except for residual effects, e.g. the Yukawa force mostly mediated by pions. In QCD, this can be understood by the confinement of all field lines (i.e. gluons) in the quark system. Since in our model, non-abelian effects are large, a similar phenomenon can be expected: Some of the gravitons are confined in the mass system (a galaxy), and the interaction between this galaxy and an outside object is smaller than expected. This conclusion is at odd to what we would conclude by explaining galaxy rotation curves with Dark Matter or with Modified Newtonian Dynamics.

It was recently observed that the expansion of the universe is accelerating [16] rather than decelerating as it was expected for a matter-dominated universe. The observation is that galaxies move away from each other at larger rates than expected. This could be explained by accounting for the zero-point energy of fields. However, there is a large discrepancy between what is needed to explain the observation and what is expected from quantum field theory, see e.g. [17] for a review. Such energy, that revived Einstein cosmological constant, is called Dark Energy. Its effect would be akin to a force repelling galaxies. However, if gravity is more feeble due to a partial confinement of gravitons in the galaxies and galaxy clusters, the difference between the expected abelian force that is assumed in calculations and the force including non-abelian effects that should be used, would appear as a repulsive force. This repulsive “force” cannot be larger than gravity itself since non-abelian effects would at most suppress the attraction of gravity outside of the mass system. Consequently, non-abelian would not explain a repulsion between galaxies, as it is observed for redshifts $z \lesssim 0.5$. However, it might explain the observations for larger redshifts. This cannot be directly predicted by our model since it cannot predict the field outside of our system.

7 Possible tests of the model

Due to the simplicity of our model, precise quantitative tests are not relevant. However, its general features and order of magnitude can be tested. We list below possible consequences of our model.

- For a linear distribution, we expect a roughly linear potential when the effective coupling
constant is large enough \((g' = \sqrt{g} \gtrsim 10^{-3})\). This regime may be testable in cluster of galaxies if the cluster is not dense.

- In the same condition, we expect a log potential for a homogeneous disk distribution. This is what is observed on spiral galaxies. Distortions of the distribution (e.g. galaxy bar and arms) should be reflected in the effective ways to account for non-abelian effects (Dark Matter distribution).

- We have no large non-abelian effects for a spherical homogeneous distribution: we should retrieve a \(\sim 1/r\) potential. A consequence is that the rotation curves of stars in roughly spherical galaxies should be closer to what is expected from a Newtonian potential, compared to the case for spiral galaxies. This constitute a possibles test for our model.

- The interaction between two spiral galaxies should be less important than the interaction of a similar system formed by two roughly spherical galaxies.

- In the past, the universe was more homogeneous, and the density fluctuations were less massive. As a consequence, the effects of Dark Energy should disappear when we reach a time when the universe was homogeneous enough. Measuring the time dependence of these effects constitutes another possible test.

- Structure formation would proceed differently since on the one hand non-baryonic Dark Matter is an essential ingredient of the current models and on the other hand, these models assume an abelian potential. This potential would underestimate gravity once non-abelian effects become important. Another test could be the investigation if structure formation without dark matter but with a stronger potential is compatible with the observed universe structure could be another possible test.

- The pertinence of models of mergers of galaxies that would not use non-baryonic Dark Matter but would use a roughly linear potential could constitute another test. We note that currents models using Cold Dark Matter largely underpredict the angular momentum of the galaxies resulting from the mergers (cooling catastrophe, see e.g. [18]) and predict too many dwarfs galaxies [19].

- Non-Abelian effects are caused by asymmetric distributions of large amounts of normal matter. Consequently, adopting the Dark Matter standpoint, baryonic matter and non-baryonic Dark Matter distributions in space should be correlated, i.e. chunks of Dark Matter should not be found away from systems of baryonic matter.
8 Summary

We have constructed a model to account for non-abelian effects in gravitation. We chose a form of Lagrangian derived from the Einstein-Hilbert Lagrangian of General Relativity under a weak field approximation. This is also the simplest possible form for a Lagrangian that contains non-linear effects and is compatible with the dimension of the gravity coupling constant. We used the natural value expected for the effective coupling constants: there are no arbitrary parameters in the model.

In spite of the model's simplicity, results are remarkably close in magnitude and shape to galaxy rotation curve data. The model explains naturally the form of the empirical effective potential used to reproduce rotation curves: non-abelian effects generate a roughly linear potential between two points. This potential is diluted to a logarithmic potential in case of an homogeneous cylindrical geometry. The Tully-Fisher relation stems naturally from such framework.

With the natural value of the effective coupling constant, no non-baryonic Dark Matter or modification of gravity is necessary. Because the actual Lagrangian for gravity is more complex than our approximation, because we used a very simple mass distribution to model a galaxy, and because the effective coupling constant used could be different from its natural value by a multiplicative factor, the possibility of contribution of non-baryonic Dark Matter cannot be ruled out. However, our model result indicates that a natural feature of gravitation, its non-abelian nature, could explain most or all of the spiral galaxy rotation curves. We also performed calculations for galaxy clusters and dwarf spiral galaxies and found a good agreement with observation.

The rise of non-abelian effects while the universe evolves from a homogeneous structure with small density fluctuations to an inhomogeneous structure with larger density fluctuations implies that the effective strength of gravity increases for most systems, as the universe ages. This may be related to the observation that the universe is presently accelerating. However, our model does not allow us to test this possibility. If non-abelian effects are responsible or contributing to the acceleration observations, another advantage compared to ad-hoc descriptions such as modified gravity would be that it is naturally compatible with empirical and strong theoretical constraints, such as fundamental symmetries, while modification of gravity respecting these constraints is very difficult. This approach also naturally solves the coincidence problem, that is the Dark Energy contribution needed to explain the present acceleration of the universe is of the same order of magnitude as the matter contribution.

Our model has been tested against known cases (Newtonian and Yukawa potentials) and it reproduces them. However, because of the approximations used and the limitations of the computation technique employed, it is possible that the non-abelian effects seen are still non-physical artifacts of the model. It is necessary to independently reproduce these effects and devise different computational approaches in order to confirm the pertinence of non-Abelian
effects to the topics of Dark Matter and Dark Energy.

References